THE UTILITY APPROACH TO MONETARY THEORY

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PART ONE

INTRODUCTION

CHAPTER I

HISTORICAL INTRODUCTION

The most striking and revolutionary development of economic thought in the last three decades is, it can be averred without hesitation, the many-sided attempt that has been made to tear down the partition built by the classical school[1] between a value theory constructed on real, non-monetary terms, and the analysis of monetary phenomena. At times, indeed, exponents of the classical school describe the monetary elements as affecting the real terms in a position of disequilibrium, or in the transition period between two different equilibria,[2] but the classical approach generally assumes that the real elements of the system offer a long-run equilibrium solution independent of the monetary conditions. Money is neutral; it is a "veil" concealing the relationships among the real terms. With some variants, the central body of the theory is essentially not so much a barter theory, as an abstraction from money and monetary functions, which furnishes the solution for equilibrium

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[1]The classical school is taken here to signify the ensemble of doctrines constructed by A. Smith and his followers up to J. S. Mill.

[2]For instance, the stimulating effect of a temporary rise in the prices is pointed out by A. Smith and J. S. Mill.

quantities and relative prices, i.e. the rates of exchange of the commodities in terms of one of them.[3] The monetary factors are logically a non-essential adjunct or appendage, usually confined to a separate part of the exposition, whose main methodological function consists in translating the relative prices into absolute, or money prices without affecting the barter solution.

The conversion from relative to money prices is brought about usually in two alternative ways by classical theorists. In one, a money price is arbitrarily attributed to the unit of the numéraire, or, what amounts to the same thing, the unit of money is defined as equal to the value of a certain quantity of the numéraire, and that rate of exchange between money and numéraire is insured on the market. The second way of determining money prices is by fixing the quantity of money at any arbitrary level and introducing the concept of velocity of money. This is defined as the ratio between the quantity of money and the total money-value of all the transactions effected during a certain period of time. Mathematically, such a definition is given by an equation, often

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[3]The commodity in terms of which all prices are reckoned is usually denoted by the Walrasian expression marchandise numéraire. L. Walras, Elements d'Economie Politique Pure (Paris: R. Pichon et R. Durand-Auzias, 1926). For a discussion of Walras' numéraire, see A. W. Marget, "The Monetary Aspect of the Walrasian System" (Journal of Political Economy, Vol. 43, No. 2, April, 1935)

called the "monetary equation." Once the quantity and the velocity of money are given, together with the value of all transactions expressed in terms of one commodity (numéraire), the monetary equation determines the money price of the numéraire and consequently of all the other commodities. In the first case, in which the money prices are determined independently of the monetary equation, the monetary equation gives the solution for the quantity of money.

Velocity is the only link that the classical school establishes between the quantity of money and its value. It is a concept peculiar to monetary theory, and renders the relationships and methods proper to the real value theory inapplicable to monetary analysis. A useful tool for certain forms of analysis, velocity has the defect of being a mere definition deduced from two other variables by tautological reasoning. For this reason it cannot furnish information about the underlying monetary relationships actually determining it, and can at best be associated with other phenomena according to observed laws of uniformity unsupported by a general deductive system. Other elements, like the interest rates, of both the monetary and the real value theories reflect this dichotomy between the two fields in the fact that they are provided by the classical and conforming works with independent theoretical bases in the real and the

monetary fields of analysis.

It can be concluded, therefore, that, although classical monetary theory is not in contradiction with real theory, it is open to general objections because of the peculiarity of its primary assumptions and of its method of analysis. Such a disunity in the concepts and method of economic theory is brought about by the classical denial of the applicability of value theory to monetary phenomena. The reasons for the departure of subsequent monetary theories from the classical approach can be found in the requirements of deductive thought for centralization and generalization of method, apart from a mere scholastic relish for symmetry and formal perfection. A closer and more significant approach to reality also imposes a break in the surface of the definition of velocity in order to penetrate the underlying forces which bring it about.

The process of thought leading to the modification or the rejection of the classical approach to money has been very slow and is still continuing at present. There have been retrogressions - a number of monetary theorists hold the classical view even now - and a variety of competitive doctrines have arisen. Some are constructive and stimulating, others often tainted with metaphysics, still others insignificant or erroneous. The trend which we intend to point out in

this array of theories is not very clear and continuous. It is actually illustrated by systems which might appear to be unrelated but which have the common characteristic of representing an evolution toward a unified and generalized economic theory.

A first advance toward a deeper insight into monetary relationships and in the application of the supply and demand method of value theory to their analysis is shown by the cash-balance doctrine. The only improvement on the cash-balance doctrine in the way of generalization and unity of method appears to be represented by the application of the marginal utility approach to the monetary field. Along with the evolution of the concept of utility, this method has undergone four different formulations which culminate in a most satisfactory version of a unified general theory of monetary and real economics. The methodological advantages of these approaches will now be briefly sketched: of the cash-balance and of the four marginal utility systems.

The cash-balance approach concentrates on the stock of money instead of on its flow. It is, therefore, more consonant than the concept of velocity of money with value theory which, in its usual statical formulation, determines the equilibrium quantities, or stocks of commodities. Its essential feature can be roughly stated by the principle that in

equilibrium the total demand for cash-balance holdings must be equal to the total quantity of money on the market.[5]

This doctrine goes back as early as the pre-classical monetary theories and reappears through the history of economic thought as vital as ever, often in new and more refined forms. It is given a mathematical interpretation by Walras, who appears to have influenced Menger and Marshall, and it is perfected and given currency by Wicksell, Hawtrey, and Keynes. Helfferich's approach of demand for and supply of money appears to be a version of the cash-balance theory.[6] The most accredited versions of the cash-balance doctrine clearly state that money holdings yield a return of convenience,

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[5]R. G. Hawtrey, in the review of L. Mises' "Theory of Money and Credit," Economic Journal, Vol. XIV, Sept. 1935, p. 512.

[6]L. Walras, op. cit., pp. 300-301 and 304-310; C. Menger, "Geld," (in "Schriften ueber Geldtheorie und Waehrungspolitik," No. 20 in Series of Reprints of Scarce Tracts in Economic and Political Science, London School of Economics, 1936), pp. 107-108; and Grundsaetze der Volkwirthschaftlehre (Wien: W. Braumueller, 1911); A. Marshall, Money, Credit, and Commerce, (IIed., London: MacMillan, 1924), pp. 43-47; and Oral Evidence before the Royal Commission on the Values of Gold and Silver, 1887, 1888, no. 9629, pp. 34-35; R. G. Hawtrey, op. cit., p. 512; and Currency and Credit, (II ed., London: Longmans, Green and Co., 1923), Ch. III; K. Helfferich, Money, (New York: Adelphi Co., 1927); J. M. Keynes, A Tract on Monetarey Reform, (New York: Harcourt, Brace and Co., 1924), pp. 83-86; and The General Theory of Employment, Interest and Money, (New York: Harcourt, Brace and Co., 1936), p. 84; D. H. Robertson, Money, (II ed., New York: Harcourt, Brace and Co., 1929), pp. 36 ff.; K. Wicksell, Lectures on Political Economy, (New York: The MacMillan Co., 1935), Vol. II, pp. 159-168. For the similarity

which can be compared to the satisfaction derived from consumption, at the margin. Among others, Marshall, Pigou, and Keynes do not leave any doubt on this point. The application of marginal analysis to the convenience of money holding is attempted also by Lavington, a supporter of the cash-balance approach. According to him the individual reaches his maximum satisfaction where the marginal return of convenience and security derived from money holding equals the marginal satisfaction of expenditure.[7] There is a wide agreement in the cash-balance approach on the fact that the satisfaction derived from holding cash does not depend on the nominal value of the amount of money, but on its purchasing power. Cash-balances, in other words, are to be reckoned in real terms, in order that the demand for them be determined.[8]

No attempt has been made by the cash-balance theorists

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of Walras' approach with the modern one, see A. W. Marget, "Leon Walras and the 'Cash-Balance Approach' to the Problem of the Value of Money," (Journal of Political Economy, Vol. 39, No. 5, Oct. 1931).

[7]F. Lavington, English Capital Market, (London: Meuthen and Co., 1921), pp. 30, 33.

[8]Marshall measures it in terms of wheat (Money, Credit, and Commerce, p. 44). See also A. C. Pigou, Essays in Applied Economics, (London: P. S. King and Son, 1923), p. 176; J. M. Keynes, op. cit.; K. Helfferich, op. cit.; K. Wicksell, Lectures on Political Economy, Vol. II, pp. 160-166. Not so explicit about the necessity of a real calculation are E. Cannan (Money, V ed., London, 1926, pp. 71-78), and R.G. Hawtrey (Currency and Credit, pp. 40-42).

to extend the methods and concepts of marginal utility to money, because, analytically, money demand differs from commodity demands, in that the quantity of money demanded is directly related to the price level. The difficulty in this application of marginal utility is unanimously found by a large number of modern authors, a part of whom belong to the cash-balance school, in an apparent logical circularity involved in the dependence of the utility of money on money prices.[9] Wicksell expresses this reproach of circular reasoning as follows: "marginal utility (of money)...depends in turn on the exchange value, or purchasing power, of the money itself, and it does not itself regulate that purchasing power."[10] According to Wicksell and to a number of other writers (Schumpeter, Wieser), money has an indirect or reflected marginal utility, derived from the goods which

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[9]H. S. Ellis furnishes the information that the objection of circularity was raised by Senior (German Monetary Theory, 1905-1933, Cambridge, Harvard Univ. Press, 1934, p. 76). See K. Wicksell, Lectures on Political Economy, Vol. II, p. 20.

L. von Mises, Theorie des Geldes und der Umlaufsmittel (Munich: Duncker und Humblot, 1912), p. 127; K. Helfferich, op. cit., Vol. II, pp. 526-527; J. Schumpeter, "Das Sozialprodukt und die Rechenpfennige" (Archiv, 44), p. 647. G. Del Vecchio criticizes the attempt to apply the theory of marginal utility to the value of money on the ground that money holding does not provide a direct service, and money utility is the utility of the goods that money can buy. (Ricerche sopra la teoria generale della moneta, Milan, 1932, ch. II).

[10]K. Wicksell, Lectures on Political Economy, Vol. II, p. 20. The underlined words are by the writer.

money purchases.[11]

Later in this section it will be indicated how the criticism of circularity may be overcome. Here it should be noted that, if correctly interpreted, the rejection of the axiom that holdings of money have a direct utility is equivalent to stating that the amount of money held does not add anything to the satisfaction derived by the individual from present consumption. According to this line of thought the only reason for money to be held at all is to be found in the money function of shifting purchasing power from present to future consumptions, consequently varying the utility derived from consumption within the constriction of the present and future expected budgets. Such a function takes care of the uses of money as a store of value; it leaves out, however, the service of facilitating transactions and of meeting contingencies in an uncertain world. The cash-balance theorists stressed the importance of the transaction service; and yet, the existence of this function of money is inconsistent with the frequent denial of a direct monetary

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[11] K. Wicksell, Lectures on Political Economy, Vol. II, p. 20; F. von Wieser, Social Economics, (New York; Adelphi Co., 1927), p. 230, pp. 262, ff.; J. Schumpeter, op. cit., p. 647. In recent works, "marginal utility of money" has been taken to mean marginal utility of income or expenditure. See R. Frisch, New Methods of Measuring Marginal Utility, (Tuebingen; Mohr, 1932); A. C. Pigou, M. Friedman, N. Georgescu - Roegen, "Marginal Utility of Money and Elasticities of Demand," (Quarterly Journal of Economics, L, 1935, pp. 532-539);

utility. Walras' mathematical device of attributing a marginal utility to money does not surmount the obstacle, as it is a fictitious concept brought forth in order to develop his formal system of equilibrium symmetrically. Walras actually did not attribute a direct utility to paper money.[12]

The application of the method of marginal utility to the monetary field has become more adequate and frutiful, together with the evolution and the perfecting of the concept of utility. In economic literature the concept of utility is rarely defined on philosophical or psychological grounds. It is usually applied as a "working concept," or a quantity of unspecified nature around which a set of intuitive hypotheses is constructed, which is indirectly verifiable by testing the plausibility of the market relations deduced from it. The economic concept of utility does not seem to have been satis-

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R. G. D. Allen, "On the Marginal Utility of Money and its Application," (Economica, Vol. 13, May 1933).

[12]Walras attributes to the goods which do not possess a direct utility, like raw materials, "services producteurs, le service de la monnaie, des raretés de convention proportionnelles a leurs prix." (Elements, p. 312). This money utility, however, is not interpreted as a mere "rareté de convention" by A. S. Marget ("The Monetary Aspect of the Walrasian System," Journal of Political Economy, Vol. 43, No. 2, April 1935, p. 157) and P. A. Samuelson (Foundations of Economic Analysis, (Cambridge: Harvard University Press, 1947, p. 118) who appraise this Walrasian approach. In this connection, Samuelson notices: "Very rightly (Walras)... reversed the commonly met dictum that 'the quantity theory should be scrapped, but the quantity equation is useful'."

factorily defined.[13] It is not essential, however, to define utility rigorously. Therefore, utility is accepted as a useful tool which may be applied as long as it helps to reach positive results in harmony with the facts.

This concept has to be qualified by a certain number of assumptions in its relation to the other economic factors, if it is to be used in a system. Different sets of assumptions may qualify it, which correspond to different analytical limitations and to various degrees of generality. Later in this work such assumptions will be carefully stated.[14] Here

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[13]In his work Historical Materialism and the Economics of K. Marx, (New York: The MacMillan Co., 1914), Benedetto Croce disputes the concept of economic choice with Pareto. In his argument, utility is not a quantity, but a quality of an object as opposed to the non-quality of the other objects. More precisely, utility is a moment, a situation of an object in its relation to the spirit. The dialectics of utility and non-utility is the dialectics of the choice, that is to will something as against not to will anything else. Evidently, Croce refers to the actual choice, the action of the individual. To this point of view, G. Tagliacozzo objects that from Croce's philosophical concept of practical choice should be distinguished the economic, or scientific, concept of theoretical choice. The latter concept has the connotation that choice is a judgment of the individual in which he orders his preferences according to their intensity, therefore taking them as quantities in an economic program. At the limit, i.e. where the range of the program should become infinitesimal, the two concepts of practical and theoretical choice are seen to coincide. G. Tagliacozzo, "Croce and the Nature of Economic Science," (Quarterly Journal of Economics, Vol. 59, May 1945). The concept of theoretical choice is similar to the concept of "value of use" as defined in the early economic literature. The concept of theoretical choice is what we call utility, according to the common use.

[14]See pp. 38-39.

it is sufficient to point out three aspects of the evolution of the concept of utility which are pertinent to monetary analysis: (1) utility as dependent on the present consumption and the services of commodities only; (2) utility as dependent on the present and future expected consumption and services of commodities; and (3) utility as dependent on commodities and on the services of money, (a) comprising future expectations, and (b) not comprising future expectations.

According to the first definition of utility, the demand for money arises only through the demand for the commodity of which money is made. Such a definition has no provision for a demand for money to be employed in its monetary use. In fact, money has no utility beside the one for its use in the arts. Furthermore, as the individual is not concerned with future consumption which has no utility for him at present, he will have no desire for storing value. Thus even the monetary function of a store of value is lacking. The statement, so frequently found in economic literature, that money is only consistent with a dynamic system, may be imputed to this definition of utility.

Formally, the process of determining static equilibrium is the same whether utility refers to the art use of money, to its monetary use, or to both, since the condition determin-

ing the amount of money held, which is that its marginal utility must be equal to the marginal utility of expenditure, does not change in any case. In comparative statics, however, the solution for the effect of a change in any price on the money holding reveals the presence or absence of an additional demand for money beside its art use, which is its monetary use; and it shows that the utility definition presently discussed is inapplicable to monetary analysis. The proof of the preceding statement will be given in a later chapter, following the exposition of the method on which it is based.[15] In his system of equilibrium equations, Pareto does not treat the marginal utility of paper money for its monetary services, for he defines utility as dependent on commodities only. This approach seems to have been followed also by von Mises, in a non-mathematical fashion.[16]

The convenience of the second definition of utility for monetary analysis will now be considered. Here utility is dependent upon the present expectations of future consumption

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[15] See Chapter III.

[16] V. Pareto, Manuel d'Economie Politique, II ed., (Paris: M. Giard, 1927), pp. 592-593 of the Appendix; L. von Mises, Theorie des Geldes und der Umlaufsmittel, p. 128. Indeed, in the text, Pareto modifies the utility of money so as to comprise also its monetary services: "la marchandise-monnaie est ophélime non seulement pour la consommation, mais aussi parce qu'elle sert a la circulation." (p. 368). But in the Appendix money's utility is the utility of the material of which money is formed.

and services of the commodities, in addition to the present consumption and services of the commodities. Holdings of money and other assets have no direct utility; in other words, they do not enter the utility function. Money and other assets will be held solely in order to modify the time-distribution of purchasing power at the disposal of the individual. If, for instance, the marginal utility of expenditure for a consumer is unsatisfactorily low at present and high at a future time, he will build up his assets at present, for the purchase of more consumption goods at the later time.

It is clear that the utility premise of this theory that money has no direct utility, is the same as the one prevailing in the cash-balance approach. Consequently, the same criticism, to which cash-balance theory was subjected from an exact point of view, can be applied here. The system takes into account merely the services of money as a store of value and leaves out the transaction and contingency services. In fact, when the distribution in time of expected purchasing power should afford the maximum of utility tothe individual, and he should have no uncertainty about his future expectations, he would hold no money, no matter how large his amount of transactions might be.

The utility-expectations approach was recently taken by Makower and Marschak as the basis for a system comprising all

assets together with money.[17] As it has been pointed out previously, the resulting theory of assets is not as general as it was in the intention of the authors, because it does not take the transactions service into account. By means of the extention of utility over time-expectations this theory modifies the system of general equilibrium of the mathematical school of Lausanne so as to comprise a theory of assets. It has the merit also of inserting money in the general picture of a theory of assets.

The approaches to monetary theory which have been examined were seen to be incomplete. The approach, according to which only the material of which money is composed has utility, neglects its monetary services totally. The qualification of the cash-balance doctrine about the applicability of the method of marginal utility and the time-expectations approach exclude the transactions and contingency services. The definition of utility as dependent on commodities and on the services of money remains to be examined. This definition may be associated with two mutually exclusive utility relationships: the relationship of utility as dependent upon present consumption and services alone and the relationship

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[17] H. Makower, J. Marschak, "Assets, Prices and Monetary Theory," (Economica, Vol. 5, 1938, pp. 261 ff.); J. Marschak, "Money and the Theory of Assets," (Econometrica, Vol. 6, No. 4, Oct., 1938).

of utility as dependent upon present consumption and services and upon the present expectations of future consumption and services. First, the assumption of money having a direct utility for the services which it performs shall be discussed; thereafter, the discussion shall turn to the advantages of the other two qualifications of the utility relationships.

The objection of circular reasoning made to the assumption of direct monetary utility has already been considered. This objection is overcome by the definition of utility as dependent on the monetary services. As recently as 1938, referring to the marginal utility of the services of money, Charles Rist wrote: "I am not aware that any scientific attempt has yet been made to apply from this standpoint the theory of marginal utility of money."[18] The amounts of the services performed by money are directly related to the quantity of money and its purchasing power. It is not an absolute amount of money-holding which is demanded, but a certain amount of purchasing power in the form of a stock of money, whatever be the number of units of money composing that stock. With a certain set of prices and a certain income the individual will desire to hold a certain amount of money. Market equilibrium is established at that price level

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[18] C. Rist, History of Monetary and Credit Theory, (New York: The MacMillan Co., 1940).

at which the total demand of the traders for cash-balances is equal to the total stock of money on the market.

The device by which utility is described as dependent on a certain quantity of purchasing power held in the form of money, rather than on money itself, is brought about by inserting the amount of money divided by a form of price-level index in the utility function. This is an important extension of the utility function. Thus, the apparently innocent assumption of direct utility of money is seen to involve the revolutionary innovation of utility being directly dependent on prices.[19] The present approach describes money holding in terms of all commodities for which it is destined to provide the services of facilitating transactions, of meeting contingency and of storing value. To be more precise, a certain part of the money stock carries out these services for each good. That portion of money is divided by the price of the good which it serves: the ratio is in the dimension of the commodity, not in the dimension of money. The commodities enter the utility function twice: once for their direct utility, a second time for the direct utility of money hold-

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[19]The high price of a commodity might make it more desirable, for an expensive object derives charms from its preciousness and adds social distinction to the owner who exhibits it. This phenomenon too is described by the introduction of the prices into the utility function. Such a refinement of the analysis can be safely neglected in a general treatment.

ing in the monetary services related to them.

Under the assumptions examined here, a change in a price has the two well-known effects given by the fundamental equation of value,[20] to which one third effect should be added. That effect operates on the amount of money demanded, due to the fact that more or less money (according to whether the price rises or falls) is necessary to carry on the same real amount of monetary services as before. Such an effect may be called the "purchasing power effect."

The main lines for the general theory of money, as sketched above, were traced by Hicks in an address made in 1934 and they were qualified and refined in his later works.[21] Hicks, however, is not very explicit about the way in which the marginal utility method could be applied to money. Very recently Samuelson made the suggestion of introducing money into the utility function. In so doing, he committed one error: he did not differentiate correctly in forming the matrix where he represents the effects of a variation in one price.[22] In an article published in 1948, Patinkin represented

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[20] E. Slutsky, "Sulla teoria del bilancio del consumatore," (Giornale degli Economisti, LI, 1915) pp. 1-26. The properties of the fundamental equation of value theory were studied also by Hotelling, Hicks, Allen, and Samuelson.

[21] J. R. Hicks, "A Suggestion for Simplifying the Theory of Money," (Economica, Vol. 2, Feb., 1935); and Value and Capital, II ed. (Oxford: Clarendon Press, 1946), expecially chs. XIII and XIX.

[22] P. A. Samuelson, Foundations of Economic Analysis, p.122.

utility as dependent on the absolute amount of money, independent of its purchasing power. This evidently described the attitude of a miser.[23]

The method of analysing the services performed by money reveals the kinship of money and other claims on wealth. Some claims perform, with different efficiency, the four main services of money: of facilitating transactions, of storing value for future consumption, of lucrative speculation, and of providing for contingencies. Some others do not have some of these functions. Undoubtedly, however, money performs the same functions as other securities. Money has perfect liquidity or saleability, and bears no risk. It also has the characteristic of being the measure of value. Considering interest on a security (that is the lucrativeness of that security) as a premium offsetting its inferior degree of liquidity and its risk, it is concluded that, from this point of view, money is the perfect security and the degree of imperfection of other securities is compensated by the interest rate charged on them. The difference among securities is thus seen to be quantitative instead of qualitative. A distinction is to be made, therefore, for each

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[23]A proof that Patinkin is wrong is that in his case the demands are not homogeneous of zero degree in the income, in all the prices, and in the amount of money initially held, as it is to be expected. D. Patinkin, "Relative Prices, Say's Law and the Demand for Money," (Econometrica, Vol. 16, No. 2, April, 1948).

security between classifications of its services: with respect to the commodities whose transactions it facilitates, or for whose future exchange it is stored; and with respect to the type of service which it performs.

The classification followed for the actual types of securities is not important as long as the properties of each type are known and securities enjoying different properties are not grouped under the same type. Three quarters of a century ago, Jevons held a similar point of view about the distinction between money and other securities. He explained that the monetary services may be distributed among different instruments and he added that it does not matter what securities we call money: "Bullion standard coin, token coin, convertible and inconvertible notes, legal tender and not legal tender, cheques of several kinds, mercantile bills, exchange bills, stock certificates, etc., are...different kinds of things."[24] In this study reference will be made to "money" for the sake of simplicity and because this term is customary. Money may be defined as a security bearing no interest, riskless and perfectly saleable, whose nominal value is invariant, or, what amounts to the same thing, that is also the measure of value. As long as these require-

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[24]S. W. Jevons, Money and the Mechanism of Exchange, (London: Paul, Trubner, Trench and Co., 1907).

ments are respected, it does not matter so much what other criterion is followed for the classification of security types. In order to establish a clear-cut rule, however, it would seem to be desirable to form a new type at the appearance of the least difference in the properties of a security. The writer fully agrees with Rist on his statement that "what matters to the economist is not so much a good definition of money, as a knowledge and an understanding of monetary phenomena." [25]

As a conclusion of this discussion of securities it is deduced that the relationships which were previously attributed to money ought to be extended to other securities. Utility is dependent also on other securities: all securities enter the utility function in the same way as money. Hicks, in his latest works, and Marschak and Makower, in their general theory of assets, reach the conclusion that pure securities or claims and commodity-assets render the same types of services.[26] Consequently, it is the task of a general theory to stress the fundamental similarity of all assets. For a necessary methodological limitation of the field of analysis, commodity-assets will not be considered, but here the competitive nature of all assets, securities

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[25] C. Rist, op. cit., p. 335.

[26] J. Marschak and H. Makower, op. cit.; J. R. Hicks, Value and Capital, chs. XIII and XIX.

and commodity-stocks should be emphasized.

The method of monetary analysis based on the assumption that a satisfaction is derived from holding money and securities is free from the criticism of neglecting one or all monetary services, which was levelled against the other approaches previously examined. Furthermore, it will be shown to have the advantage of being consistent with both a static system, with no time-expectations in the utility function and with a system comprising time-expectation in the argument of the utility function, which may be called a quasi-dynamic system, or an economic plan.[27] Obviously, the two systems are mutually exclusive, and the former is just a methodological limitation of the latter.

The static system has the advantage of a greater simplicity. It also shows that a static framework is not incompatible with monetary analysis. This was denied by the cash-balance theory and by the time-expectations approach. Similarly, the concept of money is seen to be consistent with certainty of expectations. It will appear from the results of the quasi-dynamic system that one reason for its

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[27] J. L. Mosak, General-Equilibrium Theory in International Trade, (The University of Chicago, Monograph No. 7, Cowles Commission for Research in Economics; Bloomington, The Principia Press, 1944), pp. 131-132; H. Makower, and J. Marschak, op. cit.; O. Lange, Price Flexibility and Employment, (Cowles Commission Monograph No. 8, Bloomington, 1944).

superiority lies in the distinction it makes between the transactions service and the contingency and speculation services of securities.

The high degree of generality which the present approach enjoys and the fruitful information that it yields to the analysis, together with the intuitive reasonableness of the basic hypotheses, decided for its choice in this study.

CHAPTER II

DISCUSSION OF METHOD

The preceding discussion of various approaches to monetary theory was mainly concerned with the realism and generality of the assumptions on which they are founded. Methodological considerations were only incidental. For a proper introduction to the analysis of the approach which was selected in this study such considerations are to be complemented by a more systematic discussion of method, and the field of analysis is to be carefully delimited.

The assumption that holdings of pure assets, or claims to wealth, afford a direct utility makes possible the extension of the theory of economic choice, based on the method of marginal utility, to monetary relations. By means of this theory, for any set of prices and interest rates, the quantities of commodities and securities demanded by the individual are derived from a group of utility assumptions. The quantities demanded, therefore, are the subjective elements of the system; and so are their relationships with the prices and the interest rates. The subjective elements above cannot alone determine prices and interest rates, and for this reason they must be studied in conjunction with other, objective, factors. These other factors, together

with prices and interest rates, none of which are determined by marginal utility computation alone, may be called the objective elements of the system.

A definition of the utility concept was taken as the starting point, and on this foundation, a theory of economic choice will be constructed. Hence, most of the present analysis will be based on subjective terms. This has the advantage of requiring a small number of intuitive assumptions and of offering a central unified formulation which invests the analysis with a higher degree of generality. This methodological limitation of the field to the subjective elements, however, will not affect greatly the significance of the results, for the subjective sphere is undoubtedly the most important in monetary analysis. The subjective sphere appears to be the only one where real progress can be made, as the trend of recent monetary theories seems to show. It seems to be the most controversial field where the essential points of discord among monetary doctrines are most often found.

The analysis of the objective elements will be avoided in this work. When necessary to the analysis, however, there will be selected general objective assumptions, and they will be justified on a theoretical basis sufficiently broad to permit their being accepted as "working hypotheses." Accord-

ing to the nature of the problem, an objective assumption which has a neutral effect on the system might be adopted, or different objective assumptions might be tested concurrently.

A further classification of the framework which will be examined into micro- and macro-systems is to be made here, since the subjective and objective elements play different roles in each class. In a micro-system, the analysis of marginal utility obeys the following deductive scheme: it is constituted by a subjective groundwork which determines the individual demands for commodities (and for money and other securities); which also determines the variations in these demands corresponding to variations in the other factors of the system. The summations of these individual quantities and of their variations furnish the logical basis for a theory of market stability. By means of this framework a formulation of security and commodity relationships is attained, which is organic as well as homogeneous and plural. As it has been pointed out in the first chapter, tentative systems will be set up, mutually exclusive, each one affording particular advantages to the analysis. They may, therefore, be divided into two classes: those belonging to micro-economics, and those belonging to macro-economics, i.e., those constructed on aggregates.

In the micro-systems, the objective elements play a single role. They are the prices, the interest rates, and the quantities of commodities and securities supplied to the market by the individual. They appear as given quantities, which cannot be modified by internal elements of the system, but only be external means. In mathematical terms, they are conditions of the system, or parameters, independent of the variables. Their relationships are not described by the analysis, but accepted as historical elements or data.

Objective elements are involved in the macro-systems in a more organic way: in the theory of aggregates, all elements whose amounts are not determined by the micro-systems, may be considered as objective. Objective and subjective factors have the same formal characteristics for the solution. Supply, as well as demand, contributes to the determination of market variables (prices and interest rates) as illustrated by the classical figure of the scissors' blades.

In broad lines, the objective relations are aspects of the theory of production and reflect banking and governmental policies. A systematic treatment of the theory of production is not necessary for the purpose of the analysis of aggregates, that is, to determine the conditions for market equilibrium and market stability. The latter conditions imply the study of the behavior of the demands for and supplies

of commodities and securities with respect to variations in all prices and all interest rates in the proximity of the equilibrium point. The variations in the supplies of securities are subject to three different kinds of relationships. Their behavior is studied by the theory of choice for the class of consumer loans, by the theory of production for the security issues destined to finance new capital investment, and by means of the introduction of new independent variables for the issues of governmental securities, due to independent economic policies. In particular, the variations in the supplies of the second class of securities, with respect to prices and interest rates changes within a small range around the equilibrium position, loosely represent new investment which is related to the same variables. There is no exact correspondence between enterprises' issue of securities and new investment, because the latter may also be financed from a direct allocation of a part of the profits, subtracted from the dividends.

According to the limitation of the theoretical field, supply relationships which are solutions of the production system, will be introduced in the form of mutually exclusive, "reasonable" hypotheses. The conditions will then be examined which must bind the variations of the demands alone for market stability. The alternative hypotheses which will

be suggested about the variations in the supplies of producer securities can be, therefore, interpreted as hypotheses on the shape in a small region of a schedule of efficiency of capital, in a poly-dimensional representation, where the other ordinates are the prices and the rates of interest.

Banking and governmental policies, besides production, belong to the objective elements of the theory. These policies may be divided into two main categories: the ones dictated by laws and other regulations, and the ones left to the discretion of the executive agencies. Both categories are manifestations of will and could consequently be studied by the theory of choice. The present approach is different. The theory of choice, presented here, studies only the normal economic behavior, common to all consumers and all enterprises and based on very general assumptions. A specific fact such as the decision of an executive agent affecting any aggregate value, does not yield significant information to a subjective analysis of this type. Likewise, the marginal utility tools do not permit a subjective study of the formation of a law of general economic strategy. Issues of government loans, bonds, consols, etc., the quantity of money in circulation, and the bank rates of interest are the result of economic policies. As noted above, such policies should be represented by "ulterior," independent variables and

assumptions. The hypothesis of constancy of the amount of money, both simple and fairly realistic, at least for very short periods of time, will be applied extensively in the latter part of the study.

The formal scheme, applying to uniform relations of historical facts, must also be cast in a time framework. Some economic theories dispense with the treatment of the time element, some others conceive different forms of time analysis; but in any case an economic model refers to facts which take place in time and represents a specific interpretation of relations in time. The time structure is then a criterion for the classification of economic theories, which is common to all of them. With this respect the theories can be ordered by their relative degree of generality. The most general formulations are those which extend the analysis over periods of time of any length, because they include the instantaneous or static structures as special (very restrictive) forms. They are usually designated under the name of process analysis. Static models may be considered as limited dynamic models; however, the truly dynamic, or non-static, models should be differentiated from them. Hence, only non-

static systems shall be denoted by the term dynamic. The credit for a correct definition of dynamic structures goes to R. Frisch. A dynamic structural relation is one which contains at least one variable which is related to different points of time.[1] This may be because the equation involves either time derivatives, or time integrals, or lagged and unlagged values of the variables.[2] Process analysis offers serious difficulties, because the methods suited to it which are available in pure or applied mathematics are fragmentary, yet complicated, tools not easily handled.

In agreement with Samuelson's discussion of dynamic models,[3] the process analysis frameworks may be classified into causal and causal-stochastical, historical and historic-stochastical types. A system is defined as causal at any time if its solution depends only upon the conditions at an initial point of time and the period of time elapsed since that initial point. A causal system is also known as an endogenous system. In fact, all the elements for its solution

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[1]Samuelson qualifies this statement by requiring that the variables be economically significant and that they be involved by the system at different points of time in an irremovable way. (Foundations of Economic Analysis, p. 314.) To this it should be added that the variables are to be objective. A model comprising time-expectations, for instance, does not seem to be dynamic.

[2]R. Frisch, "On the Notion of Equilibrium and Disequilibrium," (Review of Economic Studies, III, 1935-6, pp. 100-106).

[3]P. A. Samuelson, op. cit., pp. 315-316.

are contained in the initial conditions and in the structure of the set of equations. Historical models can be of the mixed endogenous-exogenous type or of the exogenous type. Such models express the impact of external causes, or factors, upon the variables under consideration. The exogenous type depends only on present time; it contains no initial conditions. The mixed type is essentially a causal system, subject to impulses from factors which are not analysed. Mathematically, the external causes, characteristic of an historical model, are represented by the appearance of time explicitly in the functionals. The stochastical complication describes a pattern of random shocks impressed on the system.[4]

A causal scheme which is transformed into an historical-causal one, albeit yielding the same results, loses generality because this can only be effectuated by leaving some variables unspecified. Quoting P. Samuelson: "Any historical system may be regarded as an 'incomplete' one."[5] It is a well known observation that frequently the usual forms of reality can only be described with a close approximation by very complicated logical schemes, while special cases of those forms, which are the product of a rare coincidence of

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[4]An example of a mixed stochastical model may be found in Schumpeter's theory of the business cycle, which stresses the accidental origin of the oscillations. (J. Schumpeter, The Theory of Economic Development, Cambridge, Harvard University Press, 1936, Ch. VI.)

[5]P. A. Samuelson, op. cit., p. 319.

factors, are described by simpler schemes. An instance of this principle is to be found in time analysis, where in order to simplify the theoretical reasoning more special cases are hypothesized by increasing the number of restrictions. It has already been noted that historical systems are less general, from a broad point of view, than causal systems. In the dynamic field, a whole gamut of increasingly special cases is obtained by limiting and simplifying the schemes. A particular case largely adopted especially in price investigations is given by linear systems of functional equations, involving first derivatives with respect to time.

The most restrictive models are in the field of static analysis. They are presented here as particular cases of the dynamical variety. It could be shown that, by relaxing some of their assumptions, they would expand into dynamic structures. The most general among the static models belong to the continuous historical variety, conceived as a cross section of time; time appearing explicitly in the parametric form. Such models are less informative than the ones to which we have already referred. They imply the restrictive hypothesis that equilibrium is reached by an instantaneous process. There is also a discontinuous historical type which applies to points of time separated by analytical periods and is based on the assumption that in reality the economic forces

operate to establish equilibrium only at those points of time. A higher degree of generality is attained by the static historical systems when their solutions still apply only to one point of time, but are affected also by subjective quantities at different points of future time, explicitly included in the systems. They are not dynamic systems, for the variables involved at different points of time are subjective; they are time-expectations. For lack of a better expression they will be referred to as quasi-dynamic systems.

The static method is often associated with the concept of stationary equilibrium. Such a tendency in economic literature is justified on theoretical grounds. The stationary equilibrium is taken to define an economy in which the variables have either a constant or a periodically repeated behaviour. Indeed, a stationary equilibrium with quantities constant in time is described as fully by a statical method as it is by a dynamic one; that is to say, a statical representation of constant stationary equilibrium is perfectly general and does not neglect any significant elements. It is needless to say that the stationary case introduces such restrictive assumptions as to give the analysis a very low grade of generality, although the method of analysis is perfectly suited to it. For an oscillatory stationary equilibrium the statical methods also appear to attain good

approximations.[6]

This sketchy classification of time structures is sufficient to define the degree of generality and the position in the total picture of economic theory of the systems that will be discussed. The attention of the reader will be first called to micro-systems of the simpler types, the stationary type and the types which we designated as discontinuous historical statical varieties. Then a quasi-dynamic micro-system, or economic plan, will be studied, where the individual equations contain variable expectations of variables and parameters in future periods of time. Expectations will be taken at first as definite and unique; in a further phase this assumption will be relaxed and a probability analysis will be introduced. The quasi-dynamic micro-systems are essentially static, because they give a solution at one point of time only. The advantage of such a representation over other static systems consists in the explicit introduction of subjective, dated factors (the time expectations), so that, although the actual time behavior is

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[6]The mathematical conditions describing a stationary state are listed by Frisch as follows: 1) all time derivatives are equal to zero, 2) the ordinates of the lagged variables and the ordinates in the time integrals have been the same in the past as they are at the point of time considered. (R. Frisch, "On the Notion of Equilibrium and Disequilibrium") It is evident that, under such conditions, any dynamic model can be transformed into a static one, without loss of generality.

not described, the result is not detached from time considerations and is, therefore, more articulate and informative. A number of factors appear in quasi-dynamic systems, which are all lumped together under the heading of "tastes" in the strictly static expressions. There are two varieties of the systems in consideration; the well-known period analysis of the Hicksian and Robertsonian type[7] and the continuous analysis. Period analysis is based on functions containing variables at different points of time taken at arbitrary equal intervals. By conceiving the intervals as infinitely small, functionals are obtained, instead of functions, and the analysis becomes continuous in time. Clearly, period analysis is a special case of continuous analysis.

The utility of micro-systems in the study of applied economics consists in the light that they throw on the underlying relationships and in the aggregate results achieved by adding together the individual quantities and variations. Static aggregate models will be constructed upon the results of micro-systems, of the same type as appear in micro-economics. Also a dynamic aggregate model constituted by a system of linear differential equations will be studied for

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[7] J. R. Hicks, Value and Capital, Ch. IX; D. H. Robertson, Banking Policy and the Price Level (London: P. S. King and Son, 1926), p. 60.

the conditions of stability. In the preceding classification such a system is defined as a special case of the pure causal model. No historical or stochastical schemes will be considered here. The present study, which attempts to extend the general theory of value to the monetary field, will make extensive use of mathematical reasoning, that proved its usefulness in marginal analysis.

The essential mathematical tools applied can be found in economic literature. Some problems require a refinement of the tools, like the special bordered Hessian giving the second order conditions for a maximum of the utility in the discontinuous analysis of the consumer's plan. All the mathematical theorems involved in the analysis are well established and well known.

PART TWO

STATIC MODELS

CHAPTER III

ANALYSIS OF THE INDIVIDUAL BEHAVIOR

A. Statement of Assumptions

The static framework appears to offer the simplest form of mathematical analysis of the choice of the individual trader. According to the approach to the concept of utility which was described above, no attempt is made at specifying its nature, but the device is applied of adopting a "working" set of assumptions about utility relationships, which comprises, for the sake of generality, only those assumptions which bear on the stability of the system. These assumptions are stated here below. Utility is a quantity, not measurable directly, that is a function of the quantities of the goods toward possession of which the economic activity of the individual is directed. Such a function is defined as unique, continous, and possessing continuous derivatives of at least the second order; some cross derivatives of the second order may be equal to zero.[1] The individual attempts to maximize

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[1]The ordinal character of utility is well known. Any monotonically increasing function of the utility function does provide the same first and second order conditions for a maximum as the original utility function. See, for instance, J. R. Hicks, Value and Capital, Mathematical Appendix.

his utility under the constriction of his limited budget. In order to attain a unique maximum solution, the condition must also be imposed that there is one and only one collection of goods which will give a restricted maximum of the utility function for any given budget.[2] The necessary and sufficient conditions imposed on the derivatives in the proximity of the maximum, or equilibrium point will be carefully stated in the course of the argument. At the present state of knowledge no empirical investigation on the shape and the characteristics of the function seems to be possible. Fortunately, direct observation furnishes two sure elements from which reliable deductions can be derived: the fact that the individual seeks the maximum of an undefined subjective quantity and a condition of stability which generally prevails in the behavior of the individual and in the market.

The money appearing in the systems of thought to be developed in this treatise is not necessarily a standard durable commodity with utility for uses other than its services in the quality of currency (as it is in the scheme of Pareto), but can be as well a convertible or inconvertible

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[2]If any point on the utility surface by a proper choice of the parameters (the budget, the prices, and the interest rates) can be made a maximum, then for a given set of parameters there is only one collection of goods that maximizes the function. The uniqueness of the restricted maximum is consequently a necessary property of the utility function, even if it is not explicitly stated, provided a maximum exists for any collection of goods.

paper money. As was indicated previously, the systems studied here include also other securities of any type. With respect to the services which they perform, money and securities are homogeneous goods. The fact that money and securities really are homogeneous is confirmed by the inconsistency among writers in placing the line of distinction between money and securities. If, on the other hand, the principle of homogeneity is accepted, it provides a concrete basis for a definition of money. Money is a security which bears no interest and conversely a security which bears no interest is classified as money. In Hicks' words: "Looked at in this way, money appears simply as the most perfect type of security; other securities are less perfect, and command a lower price because of their imperfection."[3]

Modern theory was related above as stating that securities have a direct utility, beside the indirect utility constituted by the utility of the goods which can be purchased by their exchange. The direct utility of securities arises from the fact that they are held in certain quantities. Securities and money perform their services either by being exchanged or simply by being held, but in both cases their stocks only will appear in the utility function, not their

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[3]J. R. Hicks, Value and Capital, p. 163.

flows. Neither will their velocities, i.e. the ratios between the flows and the stocks, be in the function. As a matter of fact, the fluidity and continuity of the flows have a direct bearing upon the efficiency and utility of the securities, but they are not elements of the individual's choice; they are governed by institutional and objective economic conditions. The only decision which is left to an individual in this matter is about the quantities of securities to hold within the limits of his budget. With regard to their subjective properties, securities could be compared with any durable commodity stored for any purpose. This study will not, however, consider commodity assets in order to simplify the exposition.[4] First, a static system of equilibrium for the individual trader will be considered, because its analysis possesses a great simplicity and is similar to a model often recurrent in the literature. During the discussion in the second chapter, it was shown that a

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[4]The static model will be restricted by a set of assumptions concerning the individual expectations of future incomes and future market conditions, which will be indicated at page 57. In a footnote to the same page it will be shown that under such assumptions the demands for commodity assets behave similarly to the demands for the commodities to be consumed within the analytical period. Thus, the exclusion of commodity assets is just a convenient limitation of the field of analysis, which can be readily lifted at will.

static system describes a stationary case thoroughly, and with no methodological restrictions. Stationary equilibrium is brought about by a very special concurrence of conditions, which is strikingly rare among economic phenomena. The present system may, however, be adapted to more realistic cases, by the device of representing its parameters as functions of time and of choosing a very short analytical period, within which actual oscillations would not be of importance. The individual expectations of future incomes and of future market conditions affect the system, but they do not appear explicitly in it. If the equilibrium of the system is to be determinate at all, time-expectations are to be governed by a group of assumptions. The third section of this chapter, dealing with the adjustments of the static model to changes in the parameters, contains a discussion of the behavior of this model corresponding to shifts in time-expectations. In that connection, the assumptions of stationary time-expectations will be selected for the working of the static model.

B. The Equilibrium Position

The attention of the reader is called to the utility function for the individual. For the reasons previously expressed, there will be holdings of money and other securities in the argument of the functions. Whatever the services

be, which the holding of any security performs, the amount of these services is not related to the nominal value of the security but to the amounts, which the security commands, of the goods to whose purchase it is destined. For instance, the quantity of cash that the housewife holds on hand in order to facilitate her purchase of groceries would change directly with a change in their prices, if she should decide to purchase the same amounts of groceries as she did before the change and if the marginal utility of expenditure should not vary.

The quantity held of any security may, therefore, be divided into the portions of it which are destined to facilitate transactions, to meet contingencies, and to shift purchasing power to future periods, for the purchase of each commodity. Returning to the illustration of the housewife, she holds part of her cash for the transaction and contingency services which it performs with respect to her purchase of groceries and, possibly, for saving purchasing power to the end of buying groceries in the future. Other parts of her total cash she holds for the same services with respect to each commodity entering her budget. These services with respect to each good depend on the portion of cash allocated to them, divided by the price of the good, as well as on the amount of the good which it is planned to purchase and on the

conditions of its sale, ready cash, acceptance of checks, etc. In the utility function, therefore, securities may appear separated into the portions described above, each divided by the price of the commodity to whose purchase it is assigned. The amount of the commodity to be purchased and the conditions of sale are taken care of by the structure of the function.

The utility function can be written as follows:

$$(1) \qquad u(t_o) = \phi\left(x_1(t_o), \frac{M_1(t_o)}{p_1(t_o)}, \frac{\pi_1 s_{j1}(t_o)}{p_1(t_o)}\right), \qquad \begin{matrix}(i = 1,2,\ldots,n)\\ (j = 1,2,\ldots,m)\end{matrix}$$

where u is the total utility for the individual, x_i (i = 1, 2,...,n) is the total quantity of the i-th commodity which the individual purchases on the market at the point t_o of time for his consumption during a period of time beginning at that point and of a definite length τ. The function comprises money and m other types of securities. M_i is the portion of money and s_{ji} (j = 1, 2,...,m) is the portion of the j-th kind of security, destined to the purchase of the i-th commodity with which the individual leaves the market and which he will hold during the period τ. p_i is the price of the i-th commodity and π_j is the price of the j-th security.

The utility function is subject to the constraining

relation of the budget.

$$B(t_o) = \sum_{i=1}^{n} p_i \bar{x}_i + \bar{M} + \sum_{j=1}^{m} \bar{s}_j (\pi_j + \pi_{jo} r_j)$$

(2)

$$= \sum_{i=1}^{n} p_i x_i + \sum_{i=1}^{n} M_i + \sum_{j=1}^{m} \sum_{i=1}^{n} \pi_j s_{ji} \quad ,$$

which states that the amount of purchasing power possessed by the individual is the same before and after the instantaneous market operations. The barred symbols represent the initial quantities of the goods, or the quantities of the goods supplied to the market. π_{jo} is the principal, or par value of the j-th security. r_j is the interest rate of the j-th kind of security for a period of length θ ending at the point of time t_o. The variables to be determined are the x_i's, the M_i's and the s_{ji}'s. The parameters of the system are the $\bar{x}_i$'s, $\bar{M}$, the $\bar{s}_j$'s, the p_i's, the π_j's, the π_{jo}'s, the r_j's, and the tastes of the individual expressed by the shape of his utility function. They represent the given data of the system.

The problem will be to find the first and second order conditions for the maximum value of the total utility of the individual subject to the limitation of the budget. Mathematically, this is expressed by maximizing the function defined in (1) under the constraining relation (2); that is

to say, by maximizing $\underline{Q}$ in the following expression:

$$(3)\qquad G = \phi\left(x_1, \frac{M_1}{p_1}, \frac{\pi_j s_{j1}}{p_1}\right) - \lambda\left[\sum_{i=1}^{n} p_i(x_i - \bar{x}_i) + \sum_{i=1}^{n} M_i - \bar{M} + \sum_{j=1}^{m}\left\{\pi_j \sum_{i=1}^{n} s_{ji} - \bar{s}_j(\pi_j + \pi_{jo} r_j)\right\}\right] = \text{Max.} \qquad \begin{matrix}(i=1,2,\ldots,n)\\(j=1,2,\ldots,m)\end{matrix}$$

where λ is the Lagrange multiplier.[5] The first order conditions are obtained by taking the first derivatives of (3) with respect to each variable and by setting them equal to zero. They are

$$(4)\qquad \frac{\partial \phi}{\partial x_1} - \lambda p_1 = 0 \qquad (i = 1, 2, \ldots, n)$$

$$\frac{\partial \phi}{\partial M_1} - \lambda = 0$$

$$\frac{\partial \phi}{\partial s_{j1}} - \lambda \pi_j = 0 \qquad (j = 1, 2, \ldots, m)$$

$$\sum_{i=1}^{n} p_i(x_i - \bar{x}_i) + \sum_{i=1}^{n} M_i - \bar{M} + \sum_{j=1}^{m}\left\{\pi_j \sum_{i=1}^{n} s_{ji} - \bar{s}_j(\pi_j + \pi_{jo} r_j)\right\} = 0$$

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[5] It will be noticed that income does not depend upon the choice of the individual, i.e. does not enter the utility function, in contradiction with the principle established by Senior about disutility of work. The amount of income contained in the budgetary eq. (2) is given, and its causes belong to the past, but future income could enter the present utility function. Against this point of view it can be objected that in modern economies incomes are often rigid and labor is subject to standards (number of hours, specific requirements, etc.) so that income can be taken, at least over a short period of time, as independent of the individual's choice.

The equations are the same in number as the unknowns, and consequently provide a solution determining all the unknowns. They read as follows: the utility of the marginal unit of the standard of value spent on consumption goods, which proves to be λ, is equal to the utility of the marginal unit held in the form of a security and destined to be spent on any commodity, i.e. $\frac{\partial \phi}{\partial M_i}$ and $\frac{1}{\pi_j}\frac{\partial \phi}{\partial s_{ji}}$.[6]

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[6] If it is wished to reach a result independent of the cardinal measure of utility, the analytical expressions for the indifference curves of any pair of goods can be written as follows:

$$\phi(x_1^o,\ x_2^o,\ \ldots,\ x_r,\ x_{r+1},\ \ldots,\ x_n^o;\ \frac{M_i^o}{p_1};\ \frac{\pi_j s_{ji}^o}{p_i}) = \text{const.}$$

$$\phi(x_1^o,\ x_2^o,\ \ldots,\ x_r,\ \ldots,\ x_n^o;\ \frac{M_1^o}{p_1},\ \frac{M_2^o}{p_2},\ \ldots,\ \frac{M_h}{p_h},\ \ldots,\ \frac{M_n^o}{p_n};\ \frac{\pi_j s_{ji}^o}{p_i}) = \text{const.}$$

$$(i = 1,\ 2,\ \ldots,\ n)\ (j = 1,\ 2,\ \ldots,\ m)$$

in which the variables with the superscript o have the values given by the solution of equations (4). The budgetary restrictions for the same pairs of goods are of the general form.

$$p_r x_r + p_{r+1} x_{r+1} = \text{const.}\ ;\quad p_r x_r + M_h = \text{const.}\ ;\quad \text{etc.}$$

Taking the differential of both sets of equations, it is obtained

$$\frac{\partial \phi}{\partial x_r}dx_r + \frac{\partial \phi}{\partial x_{r+1}}dx_{r+1} = 0\ ;\quad \frac{\partial \phi}{\partial x_r}dx_r + \frac{\partial \phi}{\partial M_h}dM_h = 0\ ;\quad \text{etc.}$$

And $p_r dx_r + p_{r+1}dx_{r+1} = 0$; $p_r dx_r + dM_h = 0$; etc. Whence the following solutions are immediately derived,

$$\frac{p_r}{p_{r+1}} = -\frac{dx_{r+1}}{dx_r}\ ;\quad p_r = -\frac{dM_h}{dx_r}\ ;\quad 1 = -\frac{dM_h}{dM_r}\ ;\quad \frac{\pi_q}{\pi_{q+1}} = -\frac{ds_{q+1,r}}{ds_{q,h}}\ ;\ \text{etc}$$

This framework stresses the way in which utility depends on stocks of securities and in which the services performed by the securities are related to the commodity prices. Evidently, the terms $\frac{M_i}{p_i}$ and $\frac{\prod_j s_{ji}}{p_i}$ have the same dimension as the i-th good, so that the i-th good appears in the argument of the function $(m + 2)$ times: once for its own consumption and $(m + 1)$ times for the services which money and the other m securities perform with respect to its purchase. A less informative and less suggestive set-up, but also a simpler one, consists in writing the utility function in the following form, analytically equivalent to (1)

$$(5) \qquad u = u(x_1, \frac{M}{p_1}, \frac{\pi_j s_{j1}}{p_1}) \quad , \quad (i = 1, 2, \ldots, n) \quad (j = 1, 2, \ldots, m)$$

where $M = \sum_{i=1}^{n} M_i$, and $s_j = \sum_{i=1}^{n} s_{ji}$.

At equilibrium, the marginal rate of substitution between any two commodities is inversely proportional to the ratio of their prices. The marginal rate of substitution between the portion of money held for the purchase of the h-th commodity, and the r-th commodity is equal to the price of the r-th commodity. Similarly, at equilibrium, the marginal rate of substitution between the portions of money held for the purchase of the h-th and the r-th commodities is equal to unity. Moreover, the marginal rate of substitution between the portion of the $(q + 1)$th security destined to be spent on the r-th commodity, and the portion of the q-th security held for being spent on the h-th commodity is inversely proportional to the ratio of the security prices, etc.

The budgetary constraining relation (2), then, becomes

$$B(t_o) = \sum_{i=1}^{n} p_i \bar{x}_i + \bar{M} + \sum_{j=1}^{m} \bar{s}_j(\pi_j + \pi_{jo} r_j)$$

(6)

$$= \sum_{i=1}^{n} p_i x_i + M + \sum_{j=1}^{m} \pi_j s_j \quad .$$

Define $q_i = \frac{M}{p_i}$, and $\sigma_{ji} = \frac{\pi_j s_i}{p_i}$.

The first order conditions for the maximum of the utility function (5) under the constraint (6) are

$$\frac{\partial u}{\partial x_i} = \lambda p_i \qquad (i = 1, 2, \ldots, n)$$

$$\frac{\partial u}{\partial M} = \sum_{i=1}^{n} \frac{\partial u}{\partial q_i} \frac{\partial q_i}{\partial M} = \sum_{i=1}^{n} \frac{1}{p_i} \frac{\partial u}{\partial q_i} = \lambda$$

(7)

$$\frac{\partial u}{\partial s_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial \sigma_{ji}} \frac{\partial \sigma_{ji}}{\partial s_j} = \sum_{i=1}^{n} \frac{\pi_j}{p_i} \frac{\partial u}{\partial \sigma_{ji}} = \lambda \pi_j \qquad (j=1,2,\ldots,m)$$

$$\sum_{i=1}^{n} p_i(x_i - \bar{x}_i) + M - \bar{M} + \sum_{j=1}^{m} \left\{ \pi_j s_j - \bar{s}_j(\pi_j + \pi_{jo} r_j) \right\} = 0 \quad .$$

Here again, the equations are the same in number[7] as the

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[7]They are equivalent to the conditions which we would obtain by setting the derivatives of (3) with respect to M

unknowns and, therefore, will in general furnish a solution for them. These equations state that, at equilibrium, the utility of the marginal unit of the standard of value held in the form of any security is equal to the marginal utility of expenditure, λ.

This result is to be interpreted in the light of the properties of the securities. A security is characterized by its degree of liquidity, or saleability, of lucrativity, and of risk. Securities with a very high degree of liquidity, as well as money which is perfectly liquid, are used for facilitating transactions. Bills of commerce, for instance, are held for transaction purposes; if their efficiency as means of exchange should increase, and more payments should take place by their means, their utility for the traders would increase. The contingency service is performed by these

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and s_j equal to zero. Define $q_i' = \frac{M_i}{p_i}$, and $\sigma_{ji}' = \frac{\pi_i s_{ji}}{p_i}$; the conditions are

$$\frac{\partial u}{\partial M} = \sum_{i=1}^{n} \frac{\partial u}{\partial q_i'} \frac{\partial q_i'}{\partial M_i} \frac{\partial M_i}{\partial M} = \lambda, \quad \text{and}$$

$$\frac{\partial u}{\partial s_j} = \sum_{i=1}^{n} \frac{\partial u}{\partial \sigma_{ji}'} \frac{\partial \sigma_{ji}'}{\partial s_{ji}} \frac{\partial s_{ji}}{\partial s_j} = \lambda \pi_j . \qquad (j=1,2,\ldots,m)$$

It does not seem to be possible to determine the values of the partials of M_i with respect to M, and of s_{ji} with respect to s_j.

securities and also by less liquid ones; for such a service, the lucrativeness of the asset is not overlooked. Beside these two services, securities are held for the purpose of speculation *in sensu lato*, which is here taken to signify the shift of purchasing power to the end of the period in order to increase the size of the future budgets, and to profit from the interest earned by the securities and from variations in their prices. The property of lucrativeness of the securities outweighs the property of liquidity in the speculation service. The degree of risk is an important element in all securities, but probably less important for those portions of the securities which are held for transaction purposes.

The marginal utility, therefore, of holding a security is equal to the sum of (1) the marginal utility of the security deriving from the advantage of commanding it at present, i.e. from its transaction and contingency services; and (2) the marginal utility of expenditure in the next period multiplied by a factor representing the gain in interest and the gain or loss resulting from the variation in the price of the security, i.e. the marginal utility of speculation, as it was defined above. A static analysis of the mathematical type produced here is not suited to distinguishing among the services related to liquidity and speculation, and isolating the speculation service as would be desirable in order to develop

the study of the impact of the interest rates on the system. The rates of interest appear only in the constraining relation of the budget. It would be desirable to have them appear even in the other first order conditions expressing the proportionality of the marginal utilities, because they actually affect the individual choice. This argument will be discussed in the next chapter. These objections do not refer to the formal correctness of treating stocks of securities in a static analysis of stationary and even non-stationary cases, but they cast doubt upon the usefulness of such an approach.

C. Comparative Statics

The equilibrium position is determined by the first order conditions for the restricted maximum of the utility function. The variations will now be analyzed in the equilibrium quantities demanded, corresponding to variations in the parameters of the system. For a greater simplicity in the notation, and in order to stress the fact that money is a security like any other one, s_1 will be taken to represent money. The price of this security will be, therefore, equal to one, for money is the measure of value, and its rate of interest equal to zero; or, $\Pi_1 = 1$, $r_1 = 0$. Up to this point the present system comprised money and $\underline{m}$ other securities; the total number of securities, money included, is now taken to be equal to $\underline{m}$.

1. The second order conditions for a maximum are that the quadratic form of the second derivatives of u, appropriately restricted by the budget, be negative, or the equivalent conditions that the following restricted Hessian and its principal minors are alternate in sign,[8] or

$$(8)\quad (-1)^{n+m}\begin{vmatrix} 0 & p_1 & \cdots & p_n & \pi_1 & \cdots & \pi_m \\ p_1 & u_{x_1x_1} & \cdots & u_{x_1x_n} & u_{x_1s_1} & \cdots & u_{x_1s_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_n & u_{x_nx_1} & \cdots & u_{x_nx_n} & u_{x_ns_1} & \cdots & u_{x_ns_m} \\ \pi_1 & u_{s_1x_1} & \cdots & u_{s_1x_n} & u_{s_1s_1} & \cdots & u_{s_1s_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_m & u_{s_mx_1} & \cdots & u_{s_mx_n} & u_{s_ms_1} & \cdots & u_{s_ms_m} \end{vmatrix} > 0$$

The symbol $u_{x_ix_r}$ stands for $\frac{\partial^2 u}{\partial x_r \partial x_i}$, etc.

The cross derivatives of utility with respect to a commodity and a security signify the direct change in the marginal utility of every commodity and security with respect to a unit increase in every security and commodity. Such relations may be thought to be negligible, if existent at all.

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[8] Theodore Chaundy, The Differential Calculus (Oxford: The Clarendon Press, 1935), pp. 256-259. Actually, such sufficiency conditions for a maximum are not at all necessary. There is a semi-definite case, where the Hessian vanishes. In such a case derivatives of higher order should be considered. See also pp. 247-248 and ch. X in the same work. In the following discussion this fine point will be overlooked.

Indeed, it is reasonable to assume that an increase in the individual's demand deposits affects the utility for him of his cash holding; but by how much can this increase vary the satisfaction which the individual derives from the same amount of food consumption, etc.? The only thesis in defense of the cross derivatives in question is that, *ceteris paribus*, an increased security holding might affect the "mood" of the trader and make him desire more or less the consumption of any commodity. Of course, the determinant of the sufficiency conditions for a maximum is more general, where it comprises all the terms of differentiation. The cross derivatives with respect to a commodity and a security will, however, be considered as equal to zero only when this will be necessary in order to reach definite results.

From the definition of the utility function as unique and continuous, it follows that $\frac{\partial^2 u}{\partial x_r \partial x_i} = \frac{\partial^2 u}{\partial x_i \partial x_r}$, etc., or that (8) is a symmetrical determinant. By differentiating the set of equations (7) with respect to any initial value and condition of the system, the solution is obtained for the variations caused in the equilibrium values of the variables by a change in the initial value or condition. The analysis of these variations is the analysis of the individual demand functions in the proximity of the equilibrium position. The shape of such functions is especially interesting, because

the behavior of the individual demands forms the basis for the market behavior.

ii. The discussion of the effects of changes in the parameters of the system gives relief to the element of time-expectations which actually affect all parts of the system, inasmuch as the static system takes into account stocks carried over to future times. In an economy, indeed, where there are no stocks, future expectations have no way of influencing the determination of the equilibrium position. The more realistic picture, however, of an economy where the individual is able to store purchasing power requires consideration of his more or less certain expectations of future prices, of his future income, etc. The treatment of the elements of uncertainty and risk is reserved to the sixth chapter. For the present one may adopt the convenient device of referring to "representative" time-expectations, or those certain expectations under which the individual demands would be the same as under the actual uncertain expectations.[9] This system is constructed in a time framework; it describes a market in which all transactions take place at instants of time separated by equal intervals, the analytical periods. The equilibrium situation is instantaneously reached at these points of time,

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[9]See J. R. Hicks, Value and Capital, p. 126; J. L. Mosak, op. cit., p. 129.

at which the individual provides himself with the amounts of goods which he plans to consume during the coming period, and with the amounts of securities which he intends to hold. Suppose that a parameter, explicit in the system, varies, e.g. a present price rises. It might be expected either to stay at the new level in the future, or to depart from it. If it is expected to rise still further at the end of the period, a lesser amount of the commodity will be planned to be bought in the future, and lesser amounts of securities will be demanded to be held for such a purpose. The contrary reaction will occur if the price is expected to fall. The security demands depend on time-expectations which do not appear explicitly in the system in the form of parameters. A similar reasoning, as for a change in one price, is valid for any other explicit parameter. The mathematical analysis can handle this complication, either by stating that time-expectations affect the shape of the utility function, or by inserting them into the argument of the function. The first alternative belongs to the static model, the second one is the innovation which is the basis of the economic plan that will be studied in the third chapter.

The functional relation of the static model, discussed in this study, to time-expectations is such that it is obviously unsuited to treat different variations in them. One more

limitation will have, therefore, to be imposed on the subject matter of the static system, hence the assumption will be made that all explicit parameters, prices, interest rates, and the quantities of commodities supplied to the market by the trader, are expected to be constant in the future and their present changes are expected to be permanent. This very strict assumption which should not be confused with the one of a stationary situation, greatly impairs the significance of the results, and will be dropped in the quasi-dynamic analysis of the third chapter.[10]

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[10] This assumption, however, permits a facile static treatment of commodity assets, while in its absence such assets would complicate comparative statics with serious analytical difficulties. Insert commodity stocks, denoted by the symbols y_i $(i = 1, 2, \ldots, n)$, into the static model. The equation to be maximized becomes

$$G' = \psi'(x_1, y_1, \frac{\pi_1 s_1}{p_1}) - \lambda\left[\sum_{i=1}^{n} p_i(x_i + y_i - \bar{x}_i - \bar{y}_i) + \sum_{j=1}^{m}\left\{\pi_j s_j - \bar{s}_j(\pi_j + \pi_{jo} r_j)\right\}\right] = \text{Max.}$$

$$\begin{matrix}(i = 1, 2, \ldots, n)\\ (j = 1, 2, \ldots, m)\end{matrix}$$

The necessary conditions for a maximum with respect to commodity demands are

$$\frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial y_i} = \lambda p_i \qquad (i = 1, 2, \ldots, n)$$

1. The effects of a variation in the individual supply, given by the parameters $\bar{x}_i$ and $\bar{s}_j$ will first be considered. The procedure of differentiation will be followed only for a change in the initial supply of money, $\bar{s}_1$, and the results of variations in the other parameters $\bar{x}_i$ and $\bar{s}_j$ will be stated without proof, the method applied being the same.

a. Let $\bar{s}_1$ vary. One obtains

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial \bar{s}_1} + \sum_{j=1}^{m} \pi_j \frac{\partial s_j}{\partial \bar{s}_1} = 1$$

$$- p_1 \frac{\partial \lambda}{\partial \bar{s}_1} + \sum_{i=1}^{n} u_{x_1 x_i} \frac{\partial x_i}{\partial \bar{s}_1} + \sum_{j=1}^{m} u_{x_1 s_j} \frac{\partial s_j}{\partial \bar{s}_1} = 0$$

. . . . . .

$$(9) \quad - p_n \frac{\partial \lambda}{\partial \bar{s}_1} + \sum_{i=1}^{n} u_{x_n x_i} \frac{\partial x_i}{\partial \bar{s}_1} + \sum_{j=1}^{m} u_{x_n s_j} \frac{\partial s_j}{\partial \bar{s}_1} = 0$$

$$- \pi_1 \frac{\partial \lambda}{\partial \bar{s}_1} + \sum_{i=1}^{n} u_{s_1 x_i} \frac{\partial x_i}{\partial \bar{s}_1} + \sum_{j=1}^{m} u_{s_1 s_j} \frac{\partial s_j}{\partial \bar{s}_1} = 0$$

. . . . . .

$$- \pi_m \frac{\partial \lambda}{\partial \bar{s}_1} + \sum_{i=1}^{n} u_{s_m x_i} \frac{\partial x_i}{\partial \bar{s}_1} + \sum_{j=1}^{m} u_{s_m s_j} \frac{\partial s_j}{\partial \bar{s}_1} = 0$$

The conditions for the demands for commodities destined to immediate consumption and for commodity stocks are equal, and under the assumption envisaged above, they remain equal under change in any parameter. The two sets of demands do not, therefore, need to be differentiated (distinguished).

Next proceed to define U as

$$(10) \qquad U = \begin{vmatrix} 0 & u_{x_1} & \cdots & u_{x_n} & u_{s_1} & \cdots & u_{s_m} \\ u_{x_1} & u_{x_1x_1} & \cdots & u_{x_1x_n} & u_{x_1s_1} & \cdots & u_{x_1s_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{x_n} & u_{x_nx_1} & \cdots & u_{x_nx_n} & u_{x_ns_1} & \cdots & u_{x_ns_m} \\ u_{s_1} & u_{s_1x_1} & \cdots & u_{s_1x_n} & u_{s_1s_1} & \cdots & u_{s_1s_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ u_{s_m} & u_{s_mx_1} & \cdots & u_{s_mx_n} & u_{s_ms_1} & \cdots & u_{s_ms_m} \end{vmatrix}$$

Since from equations (7) $u_{x_i} = \lambda p_i$ and $u_{s_j} = \lambda \Pi_j$, the determinant of the coefficients in (9) is equal to $-\frac{U}{\lambda^2}$. Let U_{x_i} represent the co-factor of U obtained by striking off the row and column of u_{x_i}. The effects of a variation in $\bar{s}_1$ on the unknowns determined by the system (7) are, therefore,

$$(11) \qquad \frac{\partial x_i}{\partial \bar{s}_1} = \frac{\lambda U_{x_i}}{U}, \qquad \frac{\partial s_j}{\partial \bar{s}_1} = \frac{\lambda U_{s_j}}{U}, \qquad (i = 1, 2, \ldots, n)$$

$$\frac{\partial \lambda}{\partial \bar{s}_1} = -\frac{\lambda^2 U_o}{U}. \qquad (j = 1, 2, \ldots, m)$$

Similar ratios appearing in the works on comparative statics are known as the "income effects." As the constraining rela-

tion (6) includes stocks of securities, beside income, these effects will be designated as the "budget effects." The second order conditions for the maximum of the utility function are not sufficient to determine the amount, or the sign of any single determinant ratio. Consequently, the separate effects of an increase in the initial amount of money at the disposal of the individual, upon his demands for commodities and securities, cannot be determined from the available information.

b. Let $\bar{x}_r$ vary. It is easily found that the effects on the individual demands and on the marginal utility of expenditure are

$$(12)\qquad \frac{\partial x_i}{\partial \bar{x}_r} = p_r \frac{\lambda U_{x_i}}{U} = p_r \frac{\partial x_i}{\partial \bar{s}_1}, \qquad \frac{\partial s_j}{\partial \bar{x}_r} = p_r \frac{\lambda U_{s_j}}{U} = p_r \frac{\partial s_j}{\partial \bar{s}_1},$$

$$\frac{\partial \lambda}{\partial \bar{x}_r} = -p_r \frac{\lambda^2 U_0}{U} = p_r \frac{\partial \lambda}{\partial \bar{s}_1}. \qquad (i,r = 1, 2, \ldots, n)$$

$$(j = 1, 2, \ldots, m)$$

The budgetary effects of a change in the initial supply of the commodity $\underline{x}$ are equal to the effects of a change in the initial supply of money multiplied by the price of the commodity.

c. Let $\bar{s}_q$ vary at time $(t_o - \varepsilon)$. The reader can immediately verify the set of results

$$\frac{\partial x_i}{\partial \bar{s}_q} = \frac{\lambda(\pi_q + \pi_{qo} r_q) U_{x_i}}{U} = (\pi_q + \pi_{qo} r_q) \frac{\partial x_i}{\partial \bar{s}_1},$$

$$(13) \qquad \frac{\partial s_j}{\partial \bar{s}_q} = \frac{\lambda(\pi_q + \pi_{qo} r_q) U_{s_j}}{U} = (\pi_q + \pi_{qo} r_q) \frac{\partial s_j}{\partial \bar{s}_1},$$

$$\frac{\partial \lambda}{\partial \bar{s}_q} = - \frac{\lambda^2(\pi_q + \pi_{qo} r_q) U_o}{U} = (\pi_q + \pi_{qo} r_q) \frac{\partial \lambda}{\partial \bar{s}_1}.$$

$(i = 1, 2, \ldots, n)$
$(j, q = 1, 2, \ldots, m)$

The budgetary effects of a change in the supply of the security q, purchased at the beginning of the preceding analytical period, are equal to the effects of a change in the initial supply of money multiplied by the price of the security and by the interest accrued to one unit of the security.

Beside the individual quantities supplied, three other sets of parameters affect the equilibrium solution for the quantities demanded; the commodity prices, p_i, and the security prices, π_j, together with the rates of interest, r_j. The equilibrium position is also affected by variations in the utility function, corresponding to shifts in the tastes of the individual. The effects of variations in these parameters will now be studied.

2. The effects of a change in a commodity price, p_r,

are more complicated than the ones above. Differentiating the equilibrium conditions (7) with respect to p_r, one has the following:

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} \pi_j \frac{\partial s_j}{\partial p_r} = \bar{x}_r - x_r$$

$$- p_1 \frac{\partial \lambda}{\partial p_r} + \sum_{i=1}^{n} u_{x_1 x_i} \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} u_{x_1 s_j} \frac{\partial s_j}{\partial p_r} = \sum_{j=1}^{m} u_{x_1 \sigma_{jr}} \frac{\pi_j s_j}{p_r^2}$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$- p_r \frac{\partial \lambda}{\partial p_r} + \sum_{i=1}^{n} u_{x_r x_i} \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} u_{x_r s_j} \frac{\partial s_j}{\partial p_r} = \sum_{j=1}^{m} u_{x_r \sigma_{jr}} \frac{\pi_j s_j}{p_r^2} + \lambda$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$- p_n \frac{\partial \lambda}{\partial p_r} + \sum_{i=1}^{n} u_{x_n x_i} \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} u_{x_n s_j} \frac{\partial s_j}{\partial p_r} = \sum_{j=1}^{m} u_{x_n \sigma_{jr}} \frac{\pi_j s_j}{p_r^2}$$

(14)

$$- \pi_1 \frac{\partial \lambda}{\partial p_r} + \sum_{i=1}^{n} u_{s_1 x_i} \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} u_{s_1 s_j} \frac{\partial s_j}{\partial p_r} = \frac{\pi_1}{p_r^2} \frac{\partial u}{\partial \sigma_{1r}} + \sum_{j=1}^{m} \frac{\pi_j s_j}{p_r^2} \sum_{i=1}^{n} \frac{\pi_i}{p_i} \frac{\partial^2 u}{\partial \sigma_{jr} \partial \sigma_{li}}$$

$$\cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots$$

$$- \pi_m \frac{\partial \lambda}{\partial p_r} + \sum_{i=1}^{n} u_{s_m x_i} \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} u_{s_m s_j} \frac{\partial s_j}{\partial p_r} = \frac{\pi_m}{p_r^2} \frac{\partial u}{\partial \sigma_{mr}} + \sum_{j=1}^{m} \frac{\pi_j s_j}{p_r^2} \sum_{i=1}^{n} \frac{\pi_m}{p_i} \frac{\partial^2 u}{\partial \sigma_{jr} \partial \sigma_{mi}}$$

On the basis of the discussion of the cross derivatives between commodities and securities, at page 53, the alternative of taking such derivatives as negligible in comparison to the other values is open. In the form in which it is

presently written, the set of differential expressions appears to yield little information about the shape of the demand functions. In the solution for the demand derivatives, a budget and a substitution effect appear, which will be indicated here below, but they are flanked by an amorphous sequence of terms whose signs and properties are indeterminate to all knowledge. The choice of leaving out the second cross partial derivatives of utility with respect to a commodity and a security reduces such a sequence of terms to a form more advantageous for the analysis, affecting only slightly the remaining terms. As an approximation, thus, the effect of a change in one commodity price, p_r, on the quantity of the i-th commodity demanded by the individual is given by the following expression

$$\frac{\partial x_i}{\partial p_r} = (\bar{x}_r - x_r)\frac{\lambda U_{x_i}}{U} + \frac{\lambda U_{x_r x_i}}{U} + \sum_{j=1}^{m}\left[\frac{\pi_j}{p_r^2}\frac{\partial u}{\partial \sigma_{jr}} + \frac{\pi_j}{p_r^2}\sum_{q=1}^{m}\sum_{h=1}^{n}\frac{\pi_q s_q}{p_h}\frac{\partial^2 u}{\partial \sigma_{qr}\,\partial \sigma_{jh}}\right]\frac{U_{s_j x_i}}{U}. \tag{15}$$

$$(i,r = 1, 2, \ldots, n)$$

The first term on the right-hand side of the equation represents the budgetary effect of a change in one price on the demand for the i-th good. The second term describes the substitution effect on the same demand, or the effect that

would take place alone if the variation in the real budget caused by the change in price should be exactly compensated by a direct variation in the budget. Both effects have been given much attention by recent works; they constitute the so-called fundamental equation of value theory. The other terms in the equation have been given little attention in economic literature. They are brought about by the introduction of securities in the utility function.[11]

A change in one price affects any demand, not only through the income and substitution effect, because it is a change in the budgetary restriction; but also directly, because it varies the real security holdings on which utility depends. When a price increases, the purchasing power of the securities decreases, and securities holdings perform, ceteris paribus, a lesser amount of services than before. The effects represented by the terms referring to the securities will, therefore, be denoted as the "purchasing power effects." Each bracketed expression under summation is invariant for the effect of a price change upon any variable. Their summa-

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[11]The compensating variation in the budget is interpreted in two ways: (1) such a variation as to have the same level of utility as before, i.e. moving along an indifference poly-dimensional surface; (2) a variation which allows the individual to purchase the same set of goods as before, if he should choose so. For differential changes the two interpretations are the same. For finite differences, the second variation is more convenient.

The third class of terms appear to have been derived first by C. E. V. Leser ("The Consumer's Demand for Money," Econometrica, Vol. II, April 1943, pp. 123-140).

tion will be designated by the symbols $(S_j)p_r$ and their products with the determinant ratios by $(S_j)p_r x_i$. Similarly, the substitution term will be indicated by the symbol $(X)p_r x_i$. By substituting these symbols into equation (15) and by recalling equations (11), equation (15) can be rewritten in the simpler form,

$$\frac{\partial x_i}{\partial p_r} = (\bar{x}_r - x_r)\frac{\partial x_i}{\partial \bar{s}_1} + (X)p_r x_i + \sum_{j=1}^{m}(S_j)p_r \frac{U_{s_j x_i}}{U} \tag{16}$$

$$= (\bar{x}_r - x_r)\frac{\partial x_i}{\partial \bar{s}_1} + (X)p_r x_i + \sum_{j=1}^{m}(S_j)p_r x_i \, .$$

$$(i,r = 1, 2, \ldots, n)$$

The effects of a change in p_r on the quantities of securities demanded and on the marginal utility of expenditure are, clearly, given by the analogous expressions

$$\frac{\partial s_q}{\partial p_r} = (\bar{x}_r - x_r)\frac{\partial s_q}{\partial \bar{s}_1} + (X)p_r s_q + \sum_{j=1}^{m}(S_j)p_r s_q \, , \quad \text{and} \tag{17}$$

$$\frac{\partial \lambda}{\partial p_r} = (\bar{x}_r - x_r)\frac{\partial \lambda}{\partial \bar{s}_1} - \lambda\frac{\partial x_r}{\partial \bar{s}_1} - \sum_{j=1}^{m}(S_j)p_r \frac{\partial s_j}{\partial \bar{s}_1} \, . \tag{18}$$

$$(r = 1, 2, \ldots, n)$$
$$(q = 1, 2, \ldots, m)$$

The variations of the demands for the securities, as well as the variations of the commodity demands, are divided into

three components: the budget, the substitution, and the purchasing power effects. The variation in the marginal utility of expenditure is given by three terms: the first term does not seem to suggest any economic interpretation; the second term represents a budget effect on the demand for the commodity whose price changes; the third term describes a mixed purchasing power and budget effect.

Properties of the Substitution and Purchasing Power Terms. General-equilibrium analysis yields information about the relationships among the variables only in a very small zone around the equilibrium point. Differential expressions of the type of the sets studied in this section are the only possible way to define the form of the general demand functions. More properties of the relationships of the demand functions with the prices will now be sought by examining the conditions to which the effects of a change in commodity prices are subject. The reader will notice that such conditions and in general all the information which will be reached about the demand functions are not independent or self-sustaining conditions or hypotheses, but are directly derived from the small number of hypotheses and conditions stated at the beginning of the discussion. The intuitive hypotheses, that could be challenged, concern the utility function: they

maintain its existence and continuity, the continuity of its first and second derivatives at any point, and the endeavor and ability of the individual to maximize it. The necessary and sufficient conditions for a maximum, together with the property of symmetry of the determinant U, deriving from the hypothesis of continuity of the function and of its first and second derivatives, are the only indispensable conditions of the construction. All the other restrictions concerning the demand functions can be traced back to them. The second order conditions for the restricted maximum of the utility function do not impose any specific restriction on the value and sign of the determinant ratios of the terms describing the budget effects, which are, therefore, not very informative. The second order conditions, however, attribute specific characteristics to the substitution and purchasing power terms.

The substitution terms have properties similar to those pointed out by Slutsky, Hotelling, Allen, and Hicks for systems embracing commodities only. The first six properties, listed here below, appear in the same order in which they are found in the Appendix of Hicks' Value and Capital.[12] The purchasing power terms have only one property in common with

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[12] J. R. Hicks, op. cit., pp. 310-311.

the substitution terms: the third one. The purchasing power effect of a proportionate change in all commodity prices is subject to a suggestive rule that is added to the customary six ones.

(1) Since U is symmetrical, $(X)p_r x_i = (X)p_i x_r$.[13] The purchasing power effect does not share this property. Above, it was seen that the effect of a change in one price on the variation of the demand for a commodity or a security can be divided into a budget, a substitution, and a purchasing power effect. The equality above states that, eliminating the budget and purchasing power effects, the variation in the demand for a commodity induced by a unit change in the price of another commodity is equal to the variation in the demand for the latter commodity resulting from a corresponding unit change in the price of the former. The purchasing power effects are not symmetrical because in general each variation in a commodity price affects the direct utility of security holdings in a different way.

(2) The sufficiency conditions for a maximum impose on the determinant (8) and its principal minors the conditions

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[13]The first statement of this property was made by E. Slutsky, op. cit., Slutsky's article is extensively summarized by H. Schultz (The Theory and Measurement of Demand, Chicago, 1938, Chaps. I and XIX) and R. G. D. Allen ("Professor Slutsky's Theory of Consumer's Choice," The Review of Economic Studies, Vol. 3, No. 2, Feb. 1936).

that they possess alternate signs. Since (8) is equal to $-\frac{U}{\lambda^2}$, the same conditions evidently hold for the determinant U. By shifting columns and rows in U one can transpose any $u_{x_i x_i}$ into the place of $u_{s_m s_m}$ and vice versa, and the determinant being symmetrical will still preserve its characteristics. From this, it follows that any $\frac{U_{x_i x_i}}{U}$ is negative, and, therefore, $(X)p_i x_i < 0$.[14] The purchasing power term does not have this property. The inequality means that the demand for a commodity will always decrease following an increase in the price of the same commodity, compensated so as to eliminate the budget and the purchasing power effect. The budget effect and the effect of the price change through the variation in the services yielded by security holdings might offset the substitution effect.

(3) The third property holds for the substitution as well as for the purchasing power effect. It is written

$$\sum_{i=1}^{n} p_i(X)p_r x_i + \sum_{j=1}^{m} \pi_j(X)p_r s_j = 0 \ , \quad \text{and} \quad (r = 1, 2, \ldots, n)$$

$$\sum_{i=1}^{n} p_i(s_j)p_r x_i + \sum_{q=1}^{m} \pi_q(s_j)p_r s_q = 0 \ . \quad (j = 1, 2, \ldots, m)$$

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[14] This condition was discovered by E. Slutsky and W. E. Johnson, independently. E. Slutsky, op. cit. W. E. Johnson, "The Pure Theory of Utility Curves" (Economic Journal, Vol. XXIII, No. 92, Dec. 1913), pp. 483-513.

A change in any commodity price, compensated for the budget effect, does not affect the total value of the demand for commodities and securities.[15] An intuitive way of reaching the same result is suggested by the common sense observation that a change in any price not accompanied by a change in the amount of purchasing power possessed by the individual can only cause a shift in the quantities of the goods in which he invests that amount. The mathematical proof of this property runs as follows: first, multiply equations (15) and (17), measuring the effects of the change in one commodity price, each times the price of the good whose demand variation it states. By adding these products through, the expression is obtained

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_r} + \sum_{j=1}^{m} \pi_j \frac{\partial s_j}{\partial p_r} = \frac{1}{U}(\bar{x}_r - x_r)\left[\sum_{i=1}^{n} \lambda p_i U_{x_i} + \sum_{j=1}^{m} \lambda \pi_j U_{s_j}\right]$$

$$+ \frac{1}{U}\left[\sum_{i=1}^{n} \lambda p_i U_{x_r x_i} + \sum_{j=1}^{m} \lambda \pi_j U_{x_r s_j}\right] + \frac{1}{\lambda U} \sum_{q=1}^{m}\left\{\left\{\frac{\pi_q}{p_r^2} \frac{\partial u}{\partial \sigma_{qr}}\right.\right.$$

$$\left.\left. + \frac{\pi_q}{p_r^2} \sum_{k=1}^{n} \sum_{h=1}^{n} \frac{\pi_k s_k}{p_h} \frac{\partial^2 u}{\partial \sigma_{kr} \partial \sigma_{qh}}\right\}\left\{\sum_{i=1}^{n} \lambda p_i U_{s_q x_i} + \sum_{j=1}^{m} \lambda \pi_j U_{s_q s_j}\right\}\right] .$$

$$(r = 1, 2, \ldots, n)$$

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[15] An analogous condition with respect to commodities alone was found by H. Hotelling ("Edgeworth's Taxation Paradox and the Nature of Demand and Supply Functions," Journal of Political Economy, Vol. 40, 1932, pp. 577-616).

The first term on the right-hand side of the equation represents the total budget effect and is equal to $(\bar{x}_r - x_r)$, because its coefficient is the expansion of U divided by itself, and is equal to one. The second term gives the substitution effect, but it is the expansion of a determinant with two identical rows and is, therefore, equal to zero. The coefficient of each $(S_q)p_r$ in the third term is also equal to zero, for the same reason. Consequently, the purchasing power effect through each security is null.

(4) The fourth property reads as follows, for the substitution effect,[16]

$$\beta \sum_{i=1}^{k} \sum_{r=1}^{k} p_r p_i (X) p_r x_i \qquad \text{is opposite in sign to } \beta (k < n),$$

for any value of β ; where β is arbitrary and represents a proportionate change in the $\underline{k}$ prices p_r, defining this change by means of $\underline{k}$ equations

$$\beta = \frac{dp_r}{p_r} \qquad (r = 1, 2, \ldots, k)$$

The present property is the extension of the second rule, for one commodity, to any group of commodities. It only applies

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[16]The fourth condition was obtained by Slutsky. It was found by Hotelling for a system with no budgetary restrictions. E. Slutsky, op. cit. H. Hotelling, op. cit.

to the substitution effect. A proportionate increase in the prices of any group of commodities, compensated for the budget and purchasing power effect, will cause a decrease in the total amount of expenditure on that group of commodities, and vice versa.[17]

The proof of the fourth condition is given here below. Earlier in this chapter it was noted that the sufficient conditions for a restricted maximum of the utility function, also called in the economic literature the stability conditions of the system, are that the Hessian (8) and its principal minors are alternately opposite in sign. As (8) is equal to $-\frac{U}{\lambda^2}$, the conditions are the same for U and can be

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[17]This is probably what Hicks meant, when he wrote that "if the prices of a group of goods change in the same proportion that group behaves just as if it were a single commodity." (Hicks, Value and Capital, p. 312) Mosak too makes about the same observation. (J. L. Mosak, op. cit., p. 29) Such an oversimplified statement might be misleading. In fact, all six properties apply to the group of goods treated as a single commodity. Any shift of expenditure may take place within the group itself, so that, in the case of a proportionate rise in a group of prices, some demands of commodities in that group might also increase, although the total effect on the group is negative. See also: J. R. Hicks and R. G. D. Allen, "A Reconsideration of the Theory of Value," Economica, N. S., Vol. 1, Feb. and May 1934, pp. 52-76, 196-219; especially pp. 67-68. This point will be made clear by the examination of the mathematical proof below. Similar proofs are found in Hicks' work (Value and Capital, pp. 310-311), and Mosak's analysis (op. cit., pp. 24-25). Mosak's result, written only for the total number of commodities minus one, is too limited. Furthermore, the proportionate change in a group of prices has not been expressed yet, to the writer's knowledge, in the general form reached in the proof in the text by the introduction of the multiplier.

expressed by the statement that

$$\frac{U_{x_1x_1}}{U}\ ,\quad \frac{U_{x_1x_1,x_2x_2}}{U}\ ,\ \ldots,\quad \frac{U_{x_1x_1,\ldots,x_nx_n,s_1s_1,\ldots,s_{m-1}s_{m-1}}}{U}$$

shall be alternately negative and positive. The reader should be reminded that the order in which commodities and securities are taken is arbitrary, although the same order is followed throughout this work for the sake of clarity. By a well-known theorem in determinants

$$U_{x_1x_1,\ldots,x_kx_k} = U^{1-k}\begin{vmatrix} U_{x_1x_1} & U_{x_1x_2} & \cdots & U_{x_1x_k} \\ U_{x_2x_1} & U_{x_2x_2} & \cdots & U_{x_2x_k} \\ \cdots & \cdots & \cdots & \cdots \\ U_{x_kx_1} & U_{x_kx_2} & \cdots & U_{x_kx_k} \end{vmatrix};$$

The same condition can be stated, therefore, by requiring that

$$U^{-1}U_{x_1x_1}\ ,\quad U^{-2}\begin{vmatrix} U_{x_1x_1} & U_{x_1x_2} \\ U_{x_2x_1} & U_{x_2x_2} \end{vmatrix}\ ,\ \ldots,\ U^{-k}\begin{vmatrix} U_{x_1x_1} & U_{x_1x_2} & \cdots & U_{x_1x_k} \\ U_{x_2x_1} & U_{x_2x_2} & \cdots & U_{x_2x_k} \\ \cdots & \cdots & \cdots & \cdots \\ U_{x_kx_1} & U_{x_kx_2} & \cdots & U_{x_kx_k} \end{vmatrix}$$

shall be alternately negative and positive. By substituting $\frac{U(X)p_i x_n}{\lambda}$ for $U_{x_1 x_r}$ the same conditions are obtained for the expressions

$$(19)\quad \lambda^{-1}(X)p_1x_1\ ,\ \ldots,\ \lambda^{-k}\begin{vmatrix} (X)p_1x_1 & (X)p_1x_2 & \cdots & (X)p_1x_k \\ (X)p_2x_1 & (X)p_2x_2 & \cdots & (X)p_2x_k \\ \cdots & \cdots & \cdots & \cdots \\ (X)p_kx_1 & (X)p_kx_2 & \cdots & (X)p_kx_k \end{vmatrix}$$

because λ, the marginal utility of expenditure, is positive, as it is seen from equations (7), where the marginal utility of the goods and the prices are positive.[18] Conditions (19) impose the restriction that a quadratic form

$$\sum_{i=1}^{k}\sum_{r=1}^{k} e_i e_r (X)p_r x_i \ll 0\ ,$$

for any arbitrary values of the e's not all equal to zero. In particular, the values p_r and dp_i may be substituted for e_r and e_i. The inequalities are obtained

$$\sum_{i=1}^{k}\sum_{r=1}^{k} p_i dp_r (X)p_r x_i \ll 0\ ,\quad \text{or}$$

$\beta\sum_{i=1}^{k}\sum_{r=1}^{k} p_i p_r (X)p_r x_i$ has the opposite sign of β. $(k < n)$

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[18] Expressions (19) were added by Mosak to Hicks' proof (op. cit., p. 24). It will be a useful tool for the theory of aggregates.

The budget effect for the same group is given by the first right-hand side term of the equation

$$\beta \sum_{i=1}^{k} \sum_{r=1}^{k} p_i p_r \frac{\partial x_i}{\partial p_r} = \frac{\beta\lambda}{U} \sum_{r=1}^{k} p_r(\bar{x}_r - x_r) \sum_{i=1}^{k} p_i U_{x_i}$$

$$(20) \qquad + \frac{\beta\lambda}{U} \sum_{i=1}^{k} \sum_{r=1}^{k} p_i p_r U_{x_r x_i} + \frac{\beta}{U} \sum_{r=1}^{k} \sum_{j=1}^{m} \left[\frac{\pi_j}{p_r} \frac{\partial u}{\partial \sigma_{jr}} \right.$$

$$\left. + \frac{\pi_j}{p_r} \sum_{q=1}^{m} \sum_{h=1}^{n} \frac{\pi_q s_q}{p_h} \frac{\partial^2 u}{\partial \sigma_{qr} \partial \sigma_{jh}} \right] \sum_{i=1}^{k} U_{s_j x_i} \, .$$

Its sign depends not only on $\sum_{r=1}^{k} p_r(\bar{x}_r - x_r)$ which is positive if on the whole the individual supplies the group of k goods to the market, and is negative if on the whole he takes that group of goods from the market, but also on $\frac{\lambda}{U} \sum_{i=1}^{k} p_i U_{x_i} = \sum_{i=1}^{k} p_i \frac{\partial x_i}{\partial B}$ whose sign cannot be determined by general rules. Consequently, the budget effect is indefinable in sign and can offset the negative substitution effect if $\sum_{r=1}^{k} p_r(\bar{x}_r - x_r)$ is positive and sufficiently large.

Even the purchasing power effect, described by the third term on the right-hand side of equation (20), is indefinable

in sign. The double summations of the minors do not seem to be subject to rules from the maximum conditions. Furthermore, the summation in the bracket may be of either signs. If it were negative, it would not be possible yet to determine whether it would exceed in absolute value the first element in the bracket, which is a positive quantity. Hence, the conclusion is reached that the purchasing power effect may be positive and so large as to offset the negative substitution effect.

(5) From the second and the third condition Hicks and Allen derived the fifth property, which is a condition analogous to the following one

$$\sum_{i \neq r} p_i(X) p_r x_i + \sum_{j=1}^{m} \pi_j(X) p_r s_j > 0 \quad .$$

The effect of a change in one commodity price, compensated for the budget and the purchasing power effect, upon the variations in the demands for all commodities and securities, except the commodity whose price changes, is positive.[19]

(6) From the third and the fourth rule derives the

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[19] J. R. Hicks and R. G. D. Allen, op. cit.

sixth property, expressed by

$$\beta \sum_{r=1}^{k} p_r \left[\sum_{i=k+1}^{n} p_i(X) p_r x_i + \sum_{j=1}^{m} \pi_j(X) p_r s_j \right] \text{ is of the same sign as } \beta .$$

It means that a proportionate rise in a group of commodity prices, compensated for budget and purchasing power effects, will increase the total amount of expenditure upon all the securities, and upon all the commodities whose prices did not vary.[20] In particular, where the prices of all the commodities rise proportionately, a substitution effect is shown in favor of securities, and vice versa. The statement that

$$\beta \sum_{r=1}^{n} \sum_{i=1}^{n} p_r p_i(X) p_r x_i \text{ is of the opposite sign of } \beta ,$$

together with the third rule gives

$$\beta \sum_{r=1}^{n} p_r \sum_{j=1}^{m} \pi_j(X) p_r s_j \text{ is of the same sign as } \beta .$$

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[20] The sixth rule was originally derived by Hicks for a system with commodities only. J. R. Hicks, Value and Capital, p. 311.

This result reads that the substitution effect from commodities to securities is of the same sign as the proportionate variation in the commodity prices. As the second and fourth rule do not apply to the purchasing power effect, the last two properties cannot be extended to it.

(7) The effects of a proportionate variation in all commodity prices, in the substitution and in the purchasing power term, obey an interesting rule which is listed here as the seventh property. It is stated by the following theorem: a proportionate change in all commodity prices causes only a budget effect on commodity and security demands, which results from a budget and a purchasing power effect, but it does not yield any substitution effect. This peculiar rule is explained by the nature of the consumer's demand for security holdings. The investigation of this theorem will proceed in two steps. First, it will be shown that, in an economy with no securities, both the income and the substitution effect of a proportionate change in all prices on the quantities demanded are equal to zero. Secondly, security holdings will be introduced and the consequent modifications of the problem will be examined. In this connection, the mathematical proof of the property will be given.

(a) In the absence of money and other securities, prices are defined as the rates of exchange of all commodities with

respect to one of them, the merchandise-numéraire of Walras. This is equivalent to taking the price of the numéraire as equal to one. This price, however, may be expressed in terms of a counting unit, defined as a certain fraction or multiple of the numéraire's unit of weight. In order to be expressed in the new units, the other prices also are to be multiplied by that fraction or multiple. A proportionate change in all prices is then seen to be a simple transformation of units. Inspection of the first order conditions for a maximum shows that they are invariant under such a transformation, and so is, therefore, the maximum position. In fact, calling γ the factor which performs the transformation, let p_i be the price in terms of the numéraire and γp_i be the price reckoned in the unit of account. The first order conditions are

$$\lambda = \frac{u_{x_i}}{\gamma p_i} = \frac{u_{x_r}}{p_r} \quad , \quad \sum_{i=1}^{n} \gamma p_i x_i = \sum_{i=1}^{n} \gamma p_i \bar{x}_i \quad . \quad (i,r=1,2,\ldots,n)$$

and they do not vary for any finite value of γ, for γ can be eliminated from all the equations except one, giving

$$\gamma\lambda = \frac{u_{x_i}}{p_i} = \frac{u_{x_r}}{p_r} \quad , \quad \sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i \bar{x}_i \quad . \quad (i,r=1,2,\ldots,n)$$

A proportionate change in all prices, hence, has no effect on the quantities demanded, i.e. the demands are homogeneous of zero degree in all the prices.[21] This result may be easily proved also by means of determinants, at least for small variations in the prices.[22]

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[21] Mathematically, the fact that the variables are homogeneous of zero degree in the prices indicates that there is one more parameter (the price of the numeraire in terms of the unit of account) than the number of degrees of freedom. In the equations, λ appears to be homogeneous of degree minus one in all prices. This can be immediately proved by writing the expression

$$\sum_{i=1}^{n} p_i \frac{\partial \lambda}{\partial p_i}$$

in the determinant form

$$-\frac{\lambda^2}{U'}\left[\sum_{i=1}^{n} p_i(\bar{x}_i - x_i)U'_o + \sum_{i=1}^{n} p_i U'_{x_i}\right] .$$

where U' is a determinant analogous to U for a system including only commodities. The first term is equal to zero by virtue of the budget equation. The numerator of the second term is the expansion of the determinant U', divided by λ. Clearly, the second term is equal to $-\lambda$. Consequently

$$\sum_{i=1}^{n} p_i \frac{\partial \lambda}{\partial p_i} = -\lambda .$$

Incidentally, it may be observed that Marshall's approximation of partial equilibrium analysis, for which the marginal utility of the budget (or of expenditure) is invariant with respect to a change in any price, would describe the conditions for maximum utility not restricted by the limitation of the budget, if it should be rigorously interpreted. In fact, the previous result would give $\lambda = 0$, and all the marginal utilities would be equal to zero.

[22] A different algebraic proof was given by P. A. Samuelson, Foundations of Economic Analysis, pp. 111-112.

(b) The introduction of securities in the economy provides a term of the form of a budgetary effect to describe the effect of a proportionate variation in all commodity prices. This effect is provoked by the gain or the loss which the real amounts of securities supplied undergo, according to whether prices fall or rise. The diverse amounts of securities demanded are related to commodity prices and to commodity demands. A proportionate increase in all prices curtails the purchasing power of all securities in the same proportion. Moreover, the conclusion was previously reached that, in the absence of securities, there would be no substitution effects among commodities. Therefore, no reason seems to be met for a change in the proportion among commodities and among securities; and the proportionate diminution in purchasing power is seen to provoke no substitution effect.

The theorem stated above is proved mathematically by scrutiny of the determinant expression

$$\beta \sum_{r=1}^{n} p_r \frac{\partial x_i}{\partial p_r} = \frac{\beta}{U} \left[\sum_{r=1}^{n} p_r (\bar{x}_r - x_r) \lambda U_{x_i} + \sum_{r=1}^{n} \lambda p_r U_{x_r x_i} \right.$$

$$\left. + \sum_{j=1}^{m} U_{s_j x_i} \sum_{r=1}^{n} \left\{ \frac{\pi_j}{p_r} \frac{\partial u}{\partial \sigma_{jr}} + \frac{\pi_j}{p_r} \sum_{q=1}^{m} \sum_{h=1}^{n} \frac{\pi_q s_q}{p_h} \frac{\partial^2 u}{\partial \sigma_{qr} \partial \sigma_{jh}} \right\} \right],$$

$$(i = 1, 2, \ldots, n)$$

where $\beta = \frac{dp_r}{p_r}$. But $\sum_{r=1}^{n} \frac{\pi_j}{p_r} \frac{\partial u}{\partial \sigma_{jr}} = \frac{\partial u}{\partial s_j}$,

$\pi_j \pi_q \sum_{r=1}^{n} \sum_{h=1}^{n} \frac{1}{p_r p_h} \frac{\partial^2 u}{\partial \sigma_{qr} \partial \sigma_{jh}} = \frac{\partial^2 u}{\partial s_q \partial s_j}$; and $\lambda p_r = u_{x_r}$

for equations (7). Substituting and rearranging terms, the preceding expression becomes

$$\beta \sum_{r=1}^{n} p_r \frac{\partial x_i}{\partial p_r} = \frac{\beta}{U} \left[\sum_{r=1}^{n} p_r (\bar{x}_r - x_r) \lambda U_{x_i} + \sum_{j=1}^{m} U_{s_j x_i} \sum_{q=1}^{m} s_q u_{s_j s_q} + \sum_{r=1}^{n} u_{x_r} U_{x_r x_i} + \sum_{j=1}^{m} u_{s_j} U_{s_j x_i} \right] . \qquad (i = 1, 2, \ldots, n) \tag{21}$$

The summation of the last two terms on the right-hand side is equal to zero, for it is the expansion of a determinant with two equal rows. If the sum $\sum_{q=1}^{m} s_q u_{s_q} U_{x_i}$ is added to the second term, the expansion is obtained of m determinants each with two equal columns, which are likewise equal to zero by a property in determinants. Consequently,

$$- \sum_{q=1}^{m} s_q u_{s_q} U_{x_i} = \sum_{j=1}^{m} U_{s_j x_i} \sum_{q=1}^{m} s_q u_{s_j s_q} ;$$

but $u_{s_q} = -\lambda \Pi_q$, and the budget equation (6) furnishes an expression for the first summation on the right-hand side of (21). Substituting, equation (21) is finally rewritten as

$$\beta \sum_{r=1}^{n} p_r \frac{\partial x_i}{\partial p_r} = - \frac{\beta \lambda U_{x_i}}{U} \sum_{j=1}^{m} \bar{s}_j (\pi_j + \pi_{j_0} r_j) \quad .$$

$$(i = 1, 2, \ldots, n)$$

There is left, therefore, only a term, obtained from the budget and the purchasing power terms, which has the features of a budget term and is opposite in sign to the price change. An identical proof and an analogous result apply to the analysis of the variation in the demand for any security, provoked by the proportionate price change.

It might be of interest to notice that this result affords a neat solution to the effect of a proportionate change in all commodity prices and all security holdings upon any demand. In fact, defining

$$\beta = \frac{dp_r}{p_r} = \frac{d\bar{s}_j}{\bar{s}_j} \quad , \text{ it is seen that}$$

$$\beta \sum_{j=1}^{m} \bar{s}_j \frac{\partial x_i}{\partial \bar{s}_j} = \frac{\beta \lambda U_{x_i}}{U} \sum_{j=1}^{m} \bar{s}_j (\pi_j + \pi_{j_0} r_j)$$

$$(i = 1, 2, \ldots, n)$$

which cancels out with the expression for a proportionate change in the commodity prices, giving zero as a solution. This is equivalent to saying that demands are homogeneous

of zero degree in the commodity prices and the security holdings, and it agrees with common sense. The present conclusion was reached under the assumption that the cross partial derivatives of utility with respect to any one security and any one commodity are zero. The reasonableness of the conclusion might be considered as a justification of such an assumption.

3. The effects of variations in the rates of interest still are to be examined.

a. Up to this point, only the nominal rates of interest on old securities and the dividend yield of common stocks during the previous analytical interval, figured explicitly in the budget equation. The present dividend yield of common stocks and the present and expected future market rates of interest will now be introduced into the system in an explicit form, as soon as the way is specified in which they affect the equilibrium position through the security prices. The nominal rates of interest on old securities also affect the equilibrium position indirectly through the security prices, besides appearing directly in the budget. The price of any security is a function of the security's rate of interest and of the present and expected rates of interest on securities with similar characteristics, issued at the present time and in the future.

The slightest acquaintance with the markets of investment and speculation reveals the extraordinary maze of forms of claims to wealth. Diverse legal and economic characteristics differentiate securities minutely into very small homogeneous groups.[23] Often a given security has different features from any other. Hence, the problem of expressing the relation between the price of a security and the interest rates of similar securities involves a certain amount of arbitrariness in the selection of the latter securities. This conclusion is not very satisfactory from the theoretical point of view, but it may be mitigated by the observation that securities are very close substitutes. They do not seem to experience larger fluctuations in the ratios of their prices as determined by the interest rates, than closely substitutional commodities like different qualities of wheat or sugar, do.[24] If the prices of all securities, and therefore their interest rates, have a similar behavior, a negligible discrepancy exists a fortiori among the interest rates of like securities. Consequently, the choice of the interest rates determining a security's price, among similar securities, is almost a matter

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[23] L. Chamberlain and G. W. Edwards, The Principles of Bond Investment (New York: H. Holt and Co., 1931).

[24] J. R. Hicks, Value and Capital, pp. 167-168.

of indifference.

The functions of the security prices in the interest rates present three types of structures according to whether the securities are (1) terminable loans and maturity bonds, (2) perpetual loans and consols, (3) common stocks. It appears that most securities have common characteristics with either one of these three types. Essentially, a bond is like any security bearing a fixed rate of interest, which is expected to be repaid at its face value at a definite time of maturity. A consol is typical of the security with fixed interest rate, which never reaches maturity. A common, or corporate stock shares with other non-maturing securities the common property of a variable yield. Needless to say, the above classification does not exhaust the variety of loans and mixed securities. Hybrid cases like the preferred stock, yielding a fixed dividend under the condition that there be any profit existent; like premium and discount bonds and others, are consequent to the financial policies adopted by the firm and by the government. For the sake of simplicity, such a large variety of mixed forms is to be reduced into one of the three main categories.

Denote the quantities of terminable loans and maturity bonds by the symbols s_h ($h = 1, 2, \ldots, v$); the quantities of consols by s_j ($j = v+1, v+2, \ldots, w$); and the quantities

of common stocks by s_k ($k = w+1, w+2, \ldots, m$). The bonds $\underline{h}$ have each a date of maturity τ_h. Let Π_h, Π_j, and Π_k be the prices respectively of the bond $\underline{h}$, of the consol $\underline{j}$, and of the common stock $\underline{k}$. r_h and r_j are the nominal rates of interest on bonds and consols over the analytical period. d_k and d'_k are respectively the previous and the present dividend yields of one unit of the common stock $\underline{k}$ over a period of time of the same length. R_h is also defined as the rate of interest on a bond of the type $\underline{h}$, issued at the present time and reaching maturity at time τ_h; and R_j and R_k as the interest rates on consols issued at present, similar in their properties to the $\underline{j}$ and the $\underline{k}$ securities. According to the static restriction on time-expectations mentioned above, present interest rates and dividend yields are expected to remain constant in the future.

The budget equation detailing the diverse securities assumes the form

$$\sum_{i=1}^{n} p_i(x_i - \bar{x}_i) + \sum_{h=1}^{v}\left[\pi_h s_h - \bar{s}_h(\pi_h + \pi_{ho} r_h)\right]$$
$$+ \sum_{j=v+1}^{w}\left[\pi_j s_j - \bar{s}_j(\pi_j + \pi_{jo} r_j)\right] + \sum_{k=w+1}^{m}\left[\pi_k s_k - \bar{s}_k(\pi_k + d_k)\right]$$

In order to avoid unnecessary complications, consols and

maturity bonds are assumed to be issued at par value. Bonds at a premium or at a discount are, thus, discarded. It is evident that such forms of issue are equivalent and can all be reduced to the issue at par value by means of a linear transformation.

The security prices are functions of the interest rates, which are stated by the formulas

$$\pi_h = \frac{r_h}{R_h} + \left(1 - \frac{r_h}{R_h}\right) \frac{1}{(1 + R_h)^{\tau_h - t}} \pi_{ho} \qquad (h = 1, 2, \ldots, v)$$

$$(22) \qquad \pi_j = \frac{r_j}{R_j} \pi_{jo} \qquad (j = v+1, v+2, \ldots, w)$$

$$\pi_k = \frac{d'_k}{R_k} \qquad (k = w+1, w+2, \ldots, m)$$

where t_o is the present time, and π_{ho} and π_{jo} are constants representing the principal, or par value, of the security. Obviously, the prices of presently issued securities belonging to the first two classes are invariant under a change in any interest rates and they are equal to the principals of the same securities. The reader will recall the assumption discussed at page 57, according to which all explicit para-

meters are expected to be constant in the future and their variations to be permanent. On the basis of such an assumption, the present market rates of interest are taken in equations (22) to be the same as the rates expected to govern the market in the future.

b. Variations in the market rates of interest R_h, R_j, and R_k will have the following effects on the prices of the securities issued in the past,

$$(23)\qquad \frac{\partial \pi_h}{\partial R_h} = \left[\frac{r_h}{R_h^2} \frac{1}{(1+R_h)^{\tau_h - t}} - 1) + \frac{1}{(1+R_h)^{\tau_h - t + 1}} \left\{ t - \tau_h + \frac{r_h(\tau_h - t)}{R_h} \right\} \right] \pi_{ho} \qquad (h = 1, 2, \ldots, v)$$

$$\frac{\partial \pi_j}{\partial R_j} = -\frac{r_j}{R_j^2} \pi_{jo} \qquad (j = v+1, v+2, \ldots, w)$$

$$\frac{\partial \pi_k}{\partial R_k} = -\frac{d_k'}{R_k^2} \qquad (k = w+1, w+2, \ldots, m)$$

A rise in the market rate of interest for any security is accompanied by a fall in the prices of the old securities with similar characteristics, under the assumption about time-expectations outlined above. For the same rise in the corresponding market rates of interest, the price of a maturity bond falls by a lesser percentage than the price of a

different security. The fall in the bond price is so much smaller, the nearer in the future is the maturity of the bond.

Variations in the nominal interest rates of old terminable bonds and of old consols will affect their prices in the following way,

$$\frac{\partial \pi_h}{\partial r_h} = \frac{1}{R_h}\left[1 - \frac{1}{(1 + R_h)^{T_h - t}}\right]\pi_{ho} \qquad (h = 1, 2, \ldots, v)$$

$$\frac{\partial \pi_j}{\partial r_j} = \frac{\pi_{jo}}{R_j} \qquad (j = v+1, v+2, \ldots, w)$$

The effect of a change in the present dividend yield of the common stock $\underline{k}$ on its price is given by

$$\frac{\partial \pi_k}{\partial (d_k')} = \frac{1}{R_k} \qquad (k = w+1, w+2, \ldots, m)$$

A rise in the nominal interest rates of old bonds and consols, a trivial hypothesis practically speaking, and in the present dividend yield of common stocks provokes an increase in the prices of the same securities. For an equal proportionate increase in the yield, the prices of maturity bonds will rise by a lesser percentage than the prices of different securities, and so much less, the nearer is the maturity of the bonds.

c. Substituting formulas (22) for the security prices

into the equilibrium conditions (7), and differentiating totally with respect to one rate of interest, a solution is obtained for the effect of a variation in any rate of interest upon any quantity demanded of a commodity or a security. The device of treating all securities belonging to the same type as one security will add to the simplicity of the results.

(a) The effects of a change in a market rate of interest R_h upon the quantities of commodities demanded will now be stated, except for the effects taking place through the medium of the demands for new securities, which will be treated under division (4) of this section.

$$\frac{\partial x_i}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \left[(\bar{s}_h - s_h) \frac{\lambda U_{x_i}}{U} + \sum_{q=1}^{m} \sum_{l=1}^{n} \sum_{r=1}^{n} \frac{\pi_q s_h}{p_l p_r} \frac{\partial^2 u}{\partial \sigma_{hr} \partial \sigma_{ql}} \frac{U_{s_q} x_i}{U} \right.$$

$$\left. + \left(\lambda - \sum_{l=1}^{n} \frac{1}{p_l} \frac{\partial u}{\partial \sigma_{hl}} \right) \frac{U_{s_h} x_i}{U} \right] \frac{\partial \pi_h}{\partial R_h} \qquad (i = 1, 2, \ldots, n)$$

$$(h = 1, 2, \ldots, v)$$

But $$\pi_q \sum_{l=1}^{n} \sum_{r=1}^{n} \frac{1}{p_l p_r} \frac{\partial^2 u}{\partial \sigma_{hr} \partial \sigma_{ql}} = \frac{1}{\pi_h} \frac{\partial^2 u}{\partial s_h \partial s_q}, \quad \text{and}$$

$$\sum_{l=1}^{n} \frac{\pi_h}{p_l} \frac{\partial u}{\partial \sigma_{hl}} = \frac{\partial u}{\partial s_h}.$$

Therefore, noting that the last term in the equation is equal to zero and substituting the two expressions above into the equation, it is obtained

$$\frac{\partial x_i}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \left[(\bar{s}_h - s_h)\frac{\lambda U_{x_i}}{U} - \frac{s_h}{\pi_h}\sum_{q=1}^{m} u_{s_h s_q}\frac{U_{s_q x_i}}{U} \right]\frac{\partial \pi_h}{\partial R_h} \quad .$$

$$(i = 1, 2, \ldots, n)$$

$$(h = 1, 2, \ldots, v)$$

If $\frac{u_{s_h} U_{x_i}}{U}$ is added to the terms under summation, recalling the hypothesis that the $u_{s_q x_i}$'s are equal to zero, the new summation is found to be the expansion of a determinant with two equal columns, which is equal to zero. Substituting, the result is reached,

$$\frac{\partial x_i}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \left[(\bar{s}_h - s_h)\frac{\lambda U_{x_i}}{U} + \frac{s_h}{\pi_h}\frac{\lambda \pi_h U_{x_i}}{U} \right]\frac{\partial \pi_h}{\partial R_h} = \bar{s}_h \frac{U_{x_i}}{U}\frac{\partial \pi_h}{\partial R_h} \quad .$$

$$(i = 1, 2, \ldots, n)$$

$$(h = 1, 2, \ldots, v)$$

Similarly, the effects on the demand for any security, e.g. for consols, are

$$\frac{\partial s_j}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \bar{s}_h \frac{\lambda U_{s_j}}{U}\frac{\partial \pi_j}{\partial R_j} \quad .$$

$$(h = 1, 2, \ldots, v)$$

$$(j = v+1, v+2, \ldots, w)$$

Analogous results are obtained for the effects of changes in the same market rates R_h on the securities of any other type, and for all demand effects from variations in the other market rates of interest, R_j and R_k, except for the effect of the change in any market rate on the demand for the security governed by that rate. Indeed,

$$\frac{\partial s_h}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \left[(\bar{s}_h - s_h)\frac{\lambda U_{s_h}}{U} - \frac{s_h}{\pi_h}\sum_{q=1}^{m} u_{s_h s_q}\frac{U_{s_h s_q}}{U} \right]\frac{\partial \pi_h}{\partial R_h}$$

$$(h = 1, 2, \ldots, v)$$

Adding $\frac{u_{s_h} U_{s_h}}{U}$ to the terms under summation, the expansion of the determinant U is obtained. Substituting the equivalent expression for the summation, the preceding expression becomes

$$\frac{\partial s_h}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \left[(\bar{s}_h - s_h)\frac{\lambda U_{s_h}}{U} - \frac{s_h}{\pi_h}\left(1 - \frac{u_{s_h} U_{s_h}}{U}\right) \right]\frac{\partial \pi_h}{\partial R_h},$$

$$(h = 1, 2, \ldots, v)$$

or

$$\frac{\partial s_h}{\partial \pi_h}\frac{\partial \pi_h}{\partial R_h} = \left[\bar{s}_h \frac{\lambda U_{s_h}}{U} - \frac{s_h}{\pi_h} \right]\frac{\partial \pi_h}{\partial R_h}.$$

$$(h = 1, 2, \ldots, v)$$

These conclusions state that a change in a security price, due to a variation in the market rate of interest for that type of security produces a budget effect upon the quantity of any other security demanded, which is equal to the effect upon it of a change in the initial supply of money, times the initial supply of the security whose price has varied. One more term is added to describe the effect of a change in a security price upon the demand for the same security. This term is negative; it is large if the demand for the security was large before the variation, and small if the security price was large; and vice versa.

(b) Variations in the nominal interest rates of terminable bonds and consols issued in the past are accompanied by changes in the demands for commodities and securities, which are described by similar analytical expressions. It will, therefore, be sufficient to indicate the solution for the variation in one commodity i, with respect to a change in one rate of interest, r_h, that is

$$\frac{\partial x_i}{\partial r_h} = \frac{\partial x_i}{\partial \pi_h}\cdot\frac{\partial \pi_h}{\partial r_h} + \frac{\partial x_i}{\partial B}\frac{\partial B}{\partial r_h} = \left[-\bar{s}_h \frac{\lambda U_{x_i}}{U}\right]\frac{\partial \pi_h}{\partial r_h} + \pi_{h_0}\bar{s}_h \frac{\lambda U_{x_i}}{U} \quad .$$

$$(i = 1, 2, \ldots, n)$$

$$(h = 1, 2, \ldots, v)$$

(c) The effects of a change in the present dividend yield of a corporate stock, d_k, on the quantities demanded of commodities and securities are described by the relationships stated under (a), where $\frac{\partial \Pi_k}{\partial (d)}$ is to be substituted for $\frac{\partial \Pi_k}{\partial R_k}$.

A variation in the previous dividend yield does not alter the price of the security and only affects the quantities demanded through the budgetary change which it performs. Hence,

$$\frac{\partial x_i}{\partial (d_k)} = -\bar{a}_k \frac{\lambda U_{x_i}}{U} \quad . \qquad \begin{matrix} (i = 1, 2, \ldots, n) \\ (k = w+1, w+2, \ldots, m) \end{matrix}$$

4. One more class of adjustments can take place in the system: the ones occasioned by variations in the utility function.

One way in which a change in the present market interest rate for any type of security affects the quantities demanded has already been examined. Such an effect occurs through the variation which the change in rate of interest operates in the budget, directly and through the related change in the prices of old securities belonging to this type. The present market rates of interest are the nominal rates of newly issued

securities. Hence there is a second way in which the quantities demanded are related to any such market rate, and that is through the effect of variations in this rate on the desirability of newly issued maturity bonds or consols. It will be designated as the "lucrative" effect, in the lack of a more suitable expression.

The analysis of the second effect was delayed to this place for it applies a different procedure. For the sake of simplicity, the price of a new fixed-yield security is taken to be equal to its par value which is a constant. The new security, however, is the more advantageous, the higher is its nominal rate of interest. In a static system, the relation between the demand for the new security and its rate of interest is determined by the shape of the utility function. A rise in this interest rate induces a rise in the marginal utility of the newly issued security by causing a shift in the utility function. As the analysis of such variations offers methodological difficulties, the static system is not most suited to describe the impact of interest rates on the demand for new securities. The economic plan which will be studied in the fifth chapter does not need to resort to such functional variations in order to represent the relationships presently discussed; it will, therefore, provide a more informative and exhaustive discussion of them than the static

model. Here, it will be sufficient to point out the way in which any shift in tastes modifies any demand, whether the shift is independent, or, in the case of the demand for new securities, is caused by a change in their interest rates.

Assume any shift in the preference for goods, represented by a factor α varying the marginal utility of one good. In particular, α could be taken to be the interest rate of a new fixed-yield security $\underline{h}$, which affects the marginal utility of the security. The conclusions which will be reached could then refer to the lucrative effect of a change in a present market rate. Differentiating the equilibrium conditions (7) with respect to α, it is obtained

$$\sum_{r=1}^{n} p_r \frac{\partial x_r}{\partial \alpha} + \sum_{j=1}^{m} \pi_j \frac{\partial s_j}{\partial \alpha} = 0$$

$$- p_i \frac{\partial \lambda}{\partial \alpha} + \sum_{r=1}^{n} u_{x_i x_r} \frac{\partial x_r}{\partial \alpha} + \sum_{j=1}^{m} u_{x_i s_j} \frac{\partial s_j}{\partial \alpha} = 0$$

$$- \pi_h \frac{\partial \lambda}{\partial \alpha} + \sum_{r=1}^{n} u_{s_h x_r} \frac{\partial x_r}{\partial \alpha} + \sum_{j=1}^{m} u_{s_h s_j} \frac{\partial s_j}{\partial \alpha} = - u_{s_h}$$

$$- \pi_q \frac{\partial \lambda}{\partial \alpha} + \sum_{r=1}^{n} u_{s_q x_r} \frac{\partial x_r}{\partial \alpha} + \sum_{j=1}^{m} u_{s_q s_j} \frac{\partial s_j}{\partial \alpha} = 0$$

$$(q = 1,2,h-1,h+1,\ldots,m)$$

$$(i = 1, 2, \ldots, n)$$

Solving for the rate of change in any demands and in the marginal utility of expenditure, λ, with respect to the variation in α, the following determinant solutions are attained,

$$\frac{\partial x_i}{\partial \alpha} = - u_{s_h \alpha} \frac{U_{s_h x_i}}{U} = - \frac{u_{s_h \alpha}}{\lambda} (x) p_i s_h$$

$$(24) \qquad \frac{\partial s_j}{\partial \alpha} = - u_{s_h \alpha} \frac{U_{s_h s_j}}{U}$$

$$\frac{\partial \lambda}{\partial \alpha} = u_{s_h \alpha} \frac{U_{s_h}}{U} = u_{s_h \alpha} \frac{\partial s_h}{\partial \bar{s}_1} .$$

$$\begin{matrix} (i = 1, 2, \ldots, n) \\ (j = 1, 2, \ldots, m) \end{matrix}$$

The effect of a change in the marginal utility of a security on the marginal utility of expenditure is equal to this change multiplied by the budget effect on the security. The effect of the same change on the demand for any commodity is equal to the substitution effect of the price of the commodity on the demand for the security, multiplied by $- \frac{u_{s_h \alpha}}{\lambda}$. The analogous variation in the demand for any security is expressed by a similar determinant ratio.

The following properties, therefore, which are analogous to the properties of the substitution term, hold for the

variations in any quantity demanded. For the mathematical proof of these theorems the reader is referred to the proofs of the rules of substitution, that are developed along the same lines.

(1) The demand for the good, commodity or security, whose marginal utility shifts, varies in the same direction as the shift.

(2) The effect of the change in the marginal utility of one good upon the demand for any other good is equal to the effect of a change in the marginal utility of the latter good on the demand for the former one, provided that the shift in the marginal utility of both goods be the same.

(3) A shift in any marginal utility does not affect the total value of all demands taken together.

(4) An increase (decrease) in the marginal utilities of a group of goods (commodities and securities), however uneven it may be, increases (decreases) the total amount of expenditure on that group. The proof of the fourth rule is somewhat different from the proof of the corresponding rule in the substitution effects. It is easily shown that the double summation representing the compound effect upon the total expenditure on the group is a positive definite quadratic form.

(5) The total expenditure on all the goods, except the

one whose marginal utility has increased (decreased), falls (rises).

(6) An increase (decrease) in the marginal utilities of a group of goods, however uneven it may be, decreases (increases) the total amount of expenditure on the remaining goods.

Recalling that the marginal utility of a new security varies in the same direction as the interest rate, these rules can be immediately applied to the effects of variations in the market rates of interest.

In an economy with one security of type $\underline{h}$ issued in the past, and one issued at present, the total demand effect of a change in a market rate of interest, R_h, is composed by the budget effect studied in the preceding subsection, and by the lucrative effect. Denoting the security of type $\underline{h}$ issued in the past by the symbol s_h and the security of the same type issued at present by s'_h, the total effect of a change in R_h upon any demand can be written,

$$\frac{\partial x_i}{\partial R_h} = \left[\frac{}{s_h} \frac{\lambda U_{x_i}}{U} \right] \frac{\partial \pi_h}{\partial R_h} - \frac{u_{s'_h R_h}}{\lambda} (X) p_i s'_h \quad . \qquad \begin{matrix} (i = 1, 2, \ldots, n) \\ (h = 1, 2, \ldots, v) \end{matrix}$$

Solutions of the same form hold for changes in the other market rates.

In the same economy, the total variation in the entire

amount of purchasing power spent on commodities and securities, following a change in one market rate of interest, R_h, is given by

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial R_h} + \sum_{q=1}^{m} \pi_q \frac{\partial s_q}{\partial R_h} = \left[\frac{\bar{s}_h}{U} \left(\sum_{i=1}^{n} \lambda p_i U_{x_i} + \sum_{q=1}^{m} \lambda \pi_q U_{s_q} \right) - s_h \right] \frac{\partial \pi_h}{\partial R_h} ,$$

$$(h = 1, 2, \ldots, v)$$

because the lucrative effect through the desirability of new securities is equal to zero by property (3). The expression above is equivalent to

$$\left[\frac{\bar{s}_h}{U} \left(\sum_{i=1}^{n} u_{x_i} U_{x_i} + \sum_{q=1}^{m} u_{s_q} U_{s_q} \right) - s_h \right] \frac{\partial \pi_h}{\partial R_h} .$$

$$(h = 1, 2, \ldots, v)$$

The determinant summation is the expansion of U. Hence it follows that

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial R_h} + \sum_{q=1}^{m} \pi_q \frac{\partial s_q}{\partial R_h} = \left[\bar{s}_h - s_h \right] \frac{\partial \pi_h}{\partial R_h} ,$$

$$(h = 1, 2, \ldots, v)$$

or, in words, the total variation considered here is only to be imputed to a budget effect, and is analogous to the effect of a change in one commodity price on all the quantities

demanded. Similar results are obtained for a variation in the same variables corresponding to a change in any other interest rate or in any present interest yield. An extension of these conclusions to the case of any number of securities of the same type is readily made.

The Demand Functions. Throughout this section, much attention has been given to the variations in the equilibrium quantities of commodities and securities demanded, caused by changes in any set of parameters, and to their properties or laws of uniformity. In so doing, the description was given of the structure of the individual demand functions constructed on the few assumptions carefully stated in the course of the discussion. In a rigorous fashion, the quantities of commodities and securities demanded by the individual and the marginal utility of expenditure can be formally written as functions of the parameters, as follows

$$x_i = x_i(\bar{x}_r, p_r, \bar{s}_h, \bar{s}_j, \bar{s}_k, \pi_{ho}, \pi_{jo}, r_h, r_j, R_h, R_j, R_k, d_k, d'_k, u)$$

$$s_q = s_q(\bar{x}_r, p_r, \bar{s}_h, \bar{s}_j, \bar{s}_k, \pi_{ho}, \pi_{jo}, r_h, r_j, R_h, R_j, R_k, d_k, d'_k, u)$$

$$\lambda = \lambda(\bar{x}_r, p_r, \bar{s}_h, \bar{s}_j, \bar{s}_k, \pi_{ho}, \pi_{jo}, r_h, r_j, R_h, R_j, R_k, d_k, d'_k, u)$$

$$(i,r = 1, 2, \ldots, n)$$

$$(q = 1, 2, \ldots, m)$$

$$(h = 1, 2, \ldots, v)\ (j = v+1, v+2, \ldots, w)$$

$$(k = w+1, w+2, \ldots, m)$$

π_{ho} and π_{jo} are the par value of maturity bonds and consols, respectively. The security prices do not appear among the arguments of the demand functions, because they are dependent on the interest rates, the dividend yields, and π_{ho} and π_{jo}, as indicated by (22); r_h and r_j are the nominal rates of interest on old securities, while R_h are the rates of interest on newly issued bonds; R_j and R_k are the rates on new consols, R_k referring to consols which have common characteristics with corporate stocks.

In the next chapter the market relationships, which only are of final interest to the economist, will be derived from the individual relationships established here.

D. A Special Problem: Say's Law

1. Say's law is usually enunciated with reference to market relations. The extension of this principle to individual relations requires a new definition. Say's law for the individual is defined as follows: the total market value of the demand of the individual for commodities is always

equal to the total market value of his supply of commodities. Say's law for the market will be treated at the end of the fourth chapter; in that connection there will be shown the relation between Say's law for the individual and the same assumption upon market behavior.

The following discussion aims to prove that this hypothesis for the individual is equivalent to the assumption that commodity demands are homogeneous functions of zero degree in all commodity prices. It will be subsequently shown that such assumptions are in contradiction with the assumption that the individual holds securities at all. In mathematical terms, the law for the individual is expressed by the equation

$$\sum_{i=1}^{n} p_i x_i = \sum_{i=1}^{n} p_i \bar{x}_i$$

which, together with the budget equation

$$\sum_{i=1}^{n} p_i x_i + \sum_{q=1}^{m} \pi_q s_q = \sum_{i=1}^{n} p_i \bar{x}_i + \sum_{q=1}^{m} \bar{s}_q (\pi_q + \pi_{qo} r_q) \quad ,$$

gives the corollary condition

$$\sum_{q=1}^{m} \pi_q s_q = \sum_{q=1}^{m} \bar{s}_q (\pi_q + \pi_{qo} r_q) \quad . \tag{25}$$

The equivalence of Say's assumption with the assumption of homogeneity of zero degree of commodity demands in the commodity prices[25] appears immediately from inspection of the budget equation. The assumption of homogeneity has as a consequence, the $\bar{x}$'s being constant,

$$\gamma \sum_{i=1}^{n} p_i(x_i - \bar{x}_i) + \sum_{q=1}^{m} \pi_q s_q - \bar{s}_q(\pi_q + \pi_{qo} r_q) = 0$$

which must hold for any finite value of γ. That is possible if, and only if

$$\sum_{q=1}^{m} \pi_q s_q - \bar{s}_q(\pi_q + \pi_{qo} r_q) = 0$$

In other words, Say's law for the individual implies the assumption of homogeneity, and vice versa. In fact, henceforth allusion will be made to Say's law only, treating the homogeneity relation as its necessary and sufficient corollary. It is particularly interesting to notice that quite

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[25] The equivalence, which can also be expressed by stating that one assumption is a necessary and sufficient condition for the other one, has been recently pointed out by D. Patinkin. ("Relative Prices, Say's Law, and the Demand for Money," Econometrica, Vol. 16, No. 2, April 1948; p. 153)

different conclusions are derived from the assumption for a system where no other securities appear, except money, than for one embracing various kinds of securities.

11. For the sake of argument, consider a system with only commodities and money. In the new system, equation (25) becomes

$$(26) \qquad s_1 - \bar{s}_1 = 0 \ .$$

If s_1 must remain constant, the derivative $\frac{\partial u}{\partial s_1}$ cannot be taken. Consequently, the quantity of money s_1, even if it can affect the first and second order conditions for a maximum, is indeterminate: that is, a maximum can be found for any given value of $\bar{s}_1$. The result is a succession of relative maxima with $\bar{s}_1$ as a parameter.[27] Obviously, this is a very unrealistic case and it should, therefore, be discarded. This result does not hold for the special case in which a price π_1 different from one is attributed to money, and is set equal to zero. Then, as $\frac{\partial u}{\partial s_1} = 0$, there is an alternate solution. Either s_1 has no utility at all and the amount of money held is indeterminate, or s_1 has utility and so much of it will be held, which yields all possible utility. Economically, both possibilities are trivial.

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[27]Patinkin appears to be in error when he takes the derivative concurrently with the assumption of constancy of s_1, as defined by equation (26). (op. cit., pp. 150-151).

iii. Conclusions about Say's law imposed on a system with various securities are reached by a totally different line of reasoning. Evidently equation (25) does not require that each variable in it be constant. This fact permits more positive and realistic results. Write the equation to be maximized as

$$G = u\left(x_1, \frac{\pi_q s_q}{p_1}\right) - \lambda\left[\sum_{i=1}^{n} p_i(x_i - \bar{x}_i) + \sum_{q=1}^{m}\left\{\pi_q s_q - \bar{s}_q(\pi_q + \pi_{qo} r_q)\right\}\right] - \mu\left[\sum_{q=1}^{m}\left\{\pi_q s_q - \bar{s}_q(\pi_q + \pi_{qo} r_q)\right\}\right] = \text{Max.}, \quad \begin{matrix}(i=1,2,\ldots,n)\\(q=1,2,\ldots,m)\end{matrix} \tag{27}$$

where a second constraining relation was added to take care of the new assumption. The two constraining relations together with the differentiations of the function with respect to each variable, set equal to zero, constitute the first order conditions for a maximum. The second order conditions can be stated by requiring that the determinant U', here below, and its principal minors be alternate in sign.

$$U' = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & \pi_1 & \cdots & \pi_m \\ 0 & 0 & p_1 & \cdots & p_n & \pi_1 & \cdots & \pi_m \\ 0 & p_1 & u_{x_1x_1} & \cdots & u_{x_1x_n} & u_{x_1s_1} & \cdots & u_{x_1s_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & p_n & u_{x_nx_1} & \cdots & u_{x_nx_n} & u_{x_ns_1} & \cdots & u_{x_ns_m} \\ \pi_1 & \pi_1 & u_{s_1x_1} & \cdots & u_{s_1x_n} & u_{s_1s_1} & \cdots & u_{s_1s_m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \pi_m & \pi_m & u_{s_mx_1} & \cdots & u_{s_mx_n} & u_{s_ms_1} & \cdots & u_{s_ms_m} \end{vmatrix}$$

But the minor $U'_{s_ms_m\ldots s_1s_1} = 0$, and, being a principal minor of U', contradicts the second order conditions, by which it should be different from zero. As explained in the footnote to page 53, such a condition is a very improbable one, not taken into consideration in this study. Thus, it has been shown that in the general case of a system with different types of securities Say's law for the individual is in contradiction to the assumption of maximization of the utility function.

CHAPTER IV

THEORY OF MARKET RELATIONS

A. The Equilibrium Position

1. The foregoing discussion of a stationary economy concerns the patterns of uniformity in the economic behavior of the individual consumer in a competitive market provided with n commodities and m securities. Such a detailed investigation of the individual behavior is to be justified with consideration of its usefulness in reaching conclusions about the market relationships on which alone the interest of the economist dwells. Clearly, the quantities coming into play in macroeconomics are the sums of homogeneous individual quantities, and their variations are, accordingly, the sums of the single variations taking place in microeconomics. From the character of universality that was attributed to the hypotheses regarding the individual preference functions, it follows that such aggregate variations are subject to the laws of uniformity which have been established for the individual relationships. In particular, the direction of certain effects in the micro-system is necessarily the same as for the corresponding effects in the system of the aggregates of the individual quantities over the market. Conse-

quently, the individual results may be extended to the theory of the aggregates, and furnish the most rigorous basis for it, provided that, throughout the work of the aggregate system, it be kept in mind that it still depends on the initial parameters: the individual quantities supplied, the nominal rates of interest on old securities, and the utility functions.

The problem of adding together individual quantities and their variations requires a definition of the field of validity of the preceding microanalysis and of the market over which such quantities are summed. Hitherto the analysis was concerned only with the individual consumer, with regard to whom universal laws of behavior were found. The theory of the aggregates, however, cannot fail to extend also over the entrepreneurial activity of the traders. In the introductory statements the field of analysis was limited to the theory of choice which is focussed on the subjective elements of economic activity. Accordingly, the relationships of production which are expressed by means of technical functions, and by the "automatic" economic process of allocation of the productive factors for the maximization of profit, being extraneous to the independent decisions of the individual, will not be given attention in this study. On the other hand, working hypotheses about the behavior of the quantities supplied are to be selected. The neutral hypothesis of

constant quantities supplied, on whose basis the micro-system was developed, is probably the most satisfactory. Again, use of it will be made in the aggregate relationships. An alternative assumption, commonly accepted in economic studies,[1] is the one of diminishing productivity of the factors of production, and of diminishing returns to capital investment. Its necessary corollary is that the quantity of a product supplied is directly related to its price, and the quantity of a newly issued security is directly related to its rate of interest. It is evident that such an assumption contributes to securing stable equilibrium. It appears to be superfluous to reconsider the aggregate system in the light of this second hypothesis. The reader should note, however, that the conditions for market stability which will be established on the total demand functions under the assumption of fixed supplies become somewhat too restrictive under the alternative hypothesis and consequently may be slightly relaxed.

The real difficulty in the initial framework of the theory of the aggregates appears when the financial side of the entrepreneurial operations is taken into consideration.

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[1] A clear and general statement of it is provided by J. L. Mosak, op. cit., pp. 97-98.

The enterprises hold and create large amounts of securities and hold considerable cash balances, among which, similarly to the consumers, they maintain a definite proportion. Such a proportion must be given a theoretical basis if the summations of the individual quantities of money and other securities over the consumers' and the entrepreneurs' financial accounts is to have any significance at all. There seem to be two ways to overcome this difficulty. One is to construct a theory of costs of financial operations based on the principle that a higher degree of liquidity of the assets of the enterprise pays in convenience in the form of saving of time, labor, etc., and of the possibility of taking rapid advantage of investment opportunities, but at the same time reduces the profits for the lower interest that liquid assets yield in comparison with less saleable ones. The usefulness of this approach is doubted on the ground of the flimsiness of the judgments of costs of financial operations due to liquidity conditions, although it might be accepted from a purely theoretical point of view.

The solution indicated by Hicks constitutes the second alternative.[2] Such a solution is found in the principle of subjecting the financial side of the entrepreneurial activity

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[2]J. R. Hicks, Value and Capital, pp. 242-244.

to utility considerations. It represents an extension of the usual concepts of utility, which deserves a careful qualification. For certain enterprises the financial account is not distinguishable from the private account of the entrepreneur and no line of distinction is traceable to all practical purposes. The financial account of these firms, therefore, enters the entrepreneur's utility function. Indeed, that account serves the exchange activity of the individual, and obeys the same consideration of convenience as the private account. The only difference between the two consists in the fact that one serves the exchange activity for the aim of maximizing the profit from production, the other one performs the same service for the aim of consumption. But a difference in the motives of the services is not such as to affect the considerations about the utility of the services.

Usually, however, in modern economies the account of the firm is kept separate from the private account of the entrepreneur as a matter of general practice and of legal institutions as for the companies with limited liability, the joint-stock companies, etc. The application of the principle of utility to the financial side of this class of enterprises requires a new definition of the principle itself, according to which a utility function is attributed not only to indi-

viduals, but also to "economic units" whose behavior is the result of the decisions of one or more individuals, independent of the same individuals' choice in their activity as consumers. The point made here appears to be supported by the analogy between the cost and the utility approaches to the firm's account, analogy that goes further than the formal similarity of the mathematical procedure of maximizing respectively profit and utility. The two approaches reflect essentially the same process. One balances the saving in costs of financial operations obtained by holding liquid assets with the disadvantage of a low interest rate; the other one establishes the proportions among t he quantities of assets held with different degrees of liquidity on the basis of t he convenience of liquid assets for the working of the firm, i.e. saving of costs contra the opportunity loss of interest involved in it. The utility of holding liquid securities is just an expression for the costs-saving and the gain in business opportunities. On the whole, the use of t he utility function is probably a less definite but is also a more reliable concept than the one of financial costs and gain or loss of opportunities.

There do not seem to be definite rules for the determination of the amount of dividends to be paid out by the corporation every year and the amount of profit to be invested. This

complication does not concern a stationary economy. As for more general cases, the writer shares Hicks' conclusion that the dividend policy "is a real arbitrariness, a real peculiarity of the joint stock company....we must be prepared sometimes to treat dividend policy as an independent variable."[3]

ii. According to the assumption of perfect competition, it was assumed in the micro-analysis that the commodity prices and the market rates of interest were the independent variables, determining together with the parameters of the system the quantities of goods demanded by the individual. The causality direction was defined as flowing from the commodity prices and the interest rates to the demands, not otherwise, so that the derivatives $\frac{dP_h}{dx_i}$'s, $\frac{dR_j}{dsq}$'s, etc., were taken as equal to zero. Interpreted rigorously, such an approach would not permit the determination of the commodity prices and of the interest rates by means of the aggregate equations. This difficulty is overcome by stating that the connection of causality is very weak from the individual demands to the prices and interest rates, the $\frac{dP_h}{dx_i}$'s, $\frac{dR_j}{dsq}$'s, etc., being so small that they can be safely neglected without sensibly affecting the results. The summation of the same

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[3] J. R. Hicks, op. cit., p. 244.

derivatives over all individuals is, however, a first order rate and as such it cannot be overlooked.

In the market equilibrium commodity prices and interest rates perform the function of equating the total demand to the total supply under a given set of parameters. Letting ν be the number of traders on the market, the parameters are the individual utility functions, u^{γ} ($\gamma = 1, 2, \ldots, \nu$); the quantities of commodities, $\bar{x}_r^{\gamma}$ ($r = 1, 2, \ldots, n$), and of securities $\bar{s}_q^{\gamma}$ ($q = 1, 2, \ldots, m$), supplied by the individual; the nominal rates of interest on old maturity bonds, r_h ($h = 1, 2, \ldots, v$), and on old consols, r_j ($j = v+1, \ldots, w$); the dividend yield of common stocks in the previous period, d_k, and at present, d_k' ($k = w+1, \ldots, m$). For a constant set of these parameters, the equations of market equilibrium are

$$
\begin{aligned}
X_r &= f_r(p_1, R_h, R_j, R_k) = \bar{X}_r \\
S_q &= g_q(p_1, R_h, R_j, R_k) = \bar{S}_q \quad ,
\end{aligned}
\tag{1}
$$

$$
\begin{aligned}
&(1, r = 1, 2, \ldots, n) \quad (q = 2, 3, \ldots, m) \\
&(h = 2, 3, \ldots, v) \quad (j = v+1, v+2, \ldots, w) \\
&\qquad\qquad (k = w+1, w+2, \ldots, m)
\end{aligned}
$$

where

$$
X_r = \sum_{\gamma=1}^{\nu} x_r \quad , \quad \bar{X}_r = \sum_{\gamma=1}^{\nu} \bar{x}_r \quad ,
\tag{2}
$$

$$
S_q = \sum_{\gamma=1}^{\nu} s_q \quad , \quad \bar{S}_q = \sum_{\gamma=1}^{\nu} \bar{s}_q \quad .
$$

R_h and R_j are the present market rates of interest respectively for maturity bonds and consols, and equate demand for and supply of new and old securities of the same type concurrently with the other variables in the argument of the demand functions. R_k are the market rates for consols with similar characteristics as common stocks, and equate demand for and supply of new and old common stocks and consols of the same type. The assumptions about the expectations of future interest rates made in the preceding chapter, at pages 57 and 89 still hold here. In the determination of the equilibrium conditions, all securities of the same type are combined into one. Each type of security has one rate of interest. Consequently, the number of equilibrium equations is equal to the number of variables in the argument of the demand functions.

One of equations (1) is linearly dependent on the other $(n + m - 1)$ equations and on the summation of the budget equations over all individuals. There are, therefore, $(n + m - 1)$ independent equations (1). On the basis of the previous observation about the first order magnitude of the derivatives $\frac{dP_h}{dX_i}, \frac{dR_j}{dS_q}$, etc., it is easily seen that the commodity prices and the interest rates, together with the quantities demanded, are functions of the way the total quantities supplied are distributed among the individuals,

i.e. they are functions of the individual supplies, of the utility functions, of the nominal interest rates on old fixed-yield securities, and of the dividend yields of common stocks.

Equilibrium prices and interest rates could be represented as functions of the total quantities demanded and of the individual quantities supplied. However, in the lack of further hypotheses, such quantities can only determine prices and interest rates when the total quantities demanded and supplied are equated; and at this equilibrium position commodity prices and interest rates vary only with a change in the distribution of the same total supplies among the traders, with changes in the utility functions and in the individual yields of stocks. A variation in the total supply $\bar{X}_r$ has a different effect on the p's, according to the way in which it is divided among the individual supplies $\bar{x}_r^{\gamma}$'s. Conversely, a different total supply may be distributed in such a way as to leave the set of prices unchanged. The relation of equilibrium commodity prices and interest rates with the parameters is expressed by

$$(3) \qquad p_i = F_i(\bar{x}_r^{\gamma}, \bar{s}_y^{\gamma}, r_h, r_j, d_k, d_k^i, u^{\gamma})$$

$$R_q = G_q(\bar{x}_r^{\gamma}, \bar{s}_y^{\gamma}, r_h, r_j, d_k, d_k^i, u^{\gamma}) \ .$$

$$(\gamma = 1, 2, \ldots, \nu) \ (i, r = 1, 2, \ldots, n)$$

$$(y = 1, 2, \ldots, m) \ (q = 2, 3, \ldots, m)$$

$$(h = 2, 3, \ldots, v) \ \ (j = v+1, \ldots, w) \ (k = w+1, \ldots, m)$$

In the light of these remarks it appears evident that the functions (1) should contain the parameters explicitly in the argument, and should be rewritten more generally[4]

$$X_r = X_r(p_1, R_h, R_j, R_k; \overline{x}_i^{\gamma}, \overline{s}_y^{\gamma}, r_h, r_j, d_k, d_k'; u^{\gamma}) = \overline{X}_r$$

(4)

$$S_q = S_q(p_1, R_h, R_j, R_k; \overline{x}_i^{\gamma}, \overline{s}_y^{\gamma}, r_h, r_j, d_k, d_k'; u^{\gamma}) = \overline{S}_q \ .$$

$(r,i = 1, 2, \ldots, n)$ $(y = 1, 2, \ldots, m)$ $(q = 2, 3, \ldots, m)$
$(h = 2, 3, \ldots, v)$ $(j = v+1, v+2, \ldots, w)$ $(k = w+1, w+2, \ldots, m)$

---

[4] A case in which only the total supplies should appear in equations (3) and the total demands should not be functions of the individual supplies, or

$$p_i = p_i(\overline{X}_r, \overline{S}_y, r_h, r_j, d_k, d_k', u^{\gamma})$$

$$R_q = R_q(\overline{X}_r, \overline{S}_y, r_h, r_j, d_k, d_k', u^{\gamma})$$

$(r,i = 1, 2, \ldots, n)$ $(y = 1, 2, \ldots, m)$ $(q = 2, 3, \ldots, m)$
$(h = 2, 3, \ldots, v)$ $(j = v+1, v+2, \ldots, w)$ $(k = w+1, w+2, \ldots, m)$

is rigorously stated as follows: let $\sum_{\gamma=1}^{\nu} \overline{x}_r^{\gamma} = \overline{X}_r = \text{const.}$, and

$\sum_{\gamma=1}^{\nu} \overline{s}_y^{\gamma} = \overline{S}_y = \text{const.}$, then there is seen that $\sum_{\gamma=1}^{\nu} \frac{\partial F_i}{\partial \overline{x}_r^{\gamma}} = 0$,

and $\sum_{\gamma=1}^{\nu} \frac{\partial G_q}{\partial \overline{x}_r^{\gamma}} = 0$; $\sum_{\gamma=1}^{\nu} \frac{\partial F_i}{\partial \overline{s}_y^{\gamma}} = 0$, and $\sum_{\gamma=1}^{\nu} \frac{\partial G_q}{\partial \overline{s}_y^{\gamma}} = 0$.

This is undoubtedly a very improbable case.

Functions (3) are dependent on equations (4). The unknowns are now the individual quantities demanded, in the number of $(n + m)$, the total quantities demanded, in the number of $(n + m)$, the n price ratios, and the $(m - 1)$ market rates of interest. In the micro-analysis, the individual demands were seen to be equal in number to the necessary conditions for the restricted maximization of the utility functions. The total demands are given by the $(n + m)$ equations (2). The $(n + m - 1)$ independent equations (4) may be solved for the determination of the n price ratios and of the $(m - 1)$ rates of interest. One price, generally taken as the price of money, is indeterminate. That is simply set as equal to one without any loss of generality, because the unit in which prices are computed is a matter of indifference. One independent variable, the money rate of interest, is set equal to zero.

B. Comparative Statics

Inspection of equations (4) reveals the possibility of three kinds of independent variations in the system: a change in the individual quantities supplied, a change in the yields of stocks and in the interest rates on old securities, and a shift in the utility functions. One more kind of variation may be added to these, in the form of a change in a dependent

variable, a commodity price or a present market rate of interest, induced by a change in a hitherto unconsidered independent variable, directly related to the dependent variable alone, which could be for instance a governmental policy of price fixing, a bank's discounting maneuver, etc.

Consideration will be given to the impact of a price change that is much stressed in the works of economics,[5] and also of a change in one rate of interest; then variations in the individual supplies and in the utility functions will be examined. The effects of variations in the nominal rates on old securities and in the dividend yields of common stocks at present and in the preceding period will not be considered. Such variations have been treated in the micro-system: it is left as an exercise to the interested reader to analyze these variations applying the procedure which will be indicated for the study of variations in the other parameters.

1. New parameters are now introduced into the system, which might, for instance, be independent governmental policies of price fixing, discount policies of the banks, etc. An original infinitesimal movement of p_r or R_q away from the equilibrium position, consequent to such policies,

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[5] J. R. Hicks, Value and Capital, p. 315; O. Lange, Price Flexibility and Employment, p. 92.

would necessarily occasion disequilibrium in the market and relaxation of some or all independent equations (4). The condition that the solution of equations (4) be unique, requires that at least one of the equations relaxes, but in general any number of equations (4) will cease holding. Sufficient information is not available in general so as to determine which ones will relax and to how much the disequilibrium will amount for each one of them. Differentiate equations (4) with respect to p_r, which varies in the way explained above. Equations (3) do not hold any longer, for they are valid only at the equilibrium position. In general there will be obtained, recalling that only $(n + m - 1)$ equations are independent and that the interest rate of money is equal to zero,

$$(5)\qquad \begin{aligned} \sum_{k=1}^{n} \frac{\partial X_i}{\partial p_k}\frac{dp_k}{dp_r} + \sum_{q=2}^{m} \frac{\partial X_i}{\partial R_q}\frac{dR_q}{dp_r} &= \frac{dX_i}{dp_r} \\ \sum_{k=1}^{n} \frac{\partial S_j}{\partial p_k}\frac{dp_k}{dp_r} + \sum_{q=2}^{m} \frac{\partial S_j}{\partial R_q}\frac{dR_q}{dp_r} &= \frac{dS_j}{dp_r} \end{aligned}$$

$$(i = 1, 2, \ldots, n)$$

$$(j = 2, 3, \ldots, m)$$

An analogous set of equations is obtained by differentiating equations (4) with respect to R_j. The partial derivatives

in the equations of differentiation form a determinant which will be denoted by the symbol J. The solution for a change in any price or interest rate with respect to p_r is

$$\frac{dp_k}{dp_r} = \sum_{i=1}^{n} \frac{dX_i}{dp_r} \frac{J_{x_i x_k}}{J} + \sum_{y=2}^{m} \frac{dS_y}{dp_r} \frac{J_{s_y x_k}}{J}$$

$$\frac{dR_q}{dp_r} = \sum_{i=1}^{n} \frac{dX_i}{dp_r} \frac{J_{x_i s_q}}{J} + \sum_{y=2}^{m} \frac{dS_y}{dp_r} \frac{J_{s_y s_q}}{J} \quad .$$

$$(r,k = 1, 2, \ldots, n) \quad (q = 2, 3, \ldots, m)$$

Again, an analogous solution is provided for a change in any price or interest rate with respect to R_q. Evidently, the solution depends on the values of the right-hand side members of the equations of differentiation, which represent the amount of disequilibrium in each total demand and are not determinable. If another price, p_h, is allowed to vary after a change in another independent parameter of the type of an independent policy, etc.; and by chance the terms on the right-hand side of equations (5) are the same as for a change in p_r, then

$$\frac{dp_k}{dp_r} = \frac{dp_k}{dp_h} \quad , \text{ and } \quad \frac{dR_q}{dp_r} = \frac{dR_q}{dp_h} \quad .$$

$$(k,h,r = 1, 2, \ldots, n) \quad (q = 2, 3, \ldots, m)$$

By a similar reasoning and under the same hypothesis, it is immediately noticed that

$$\frac{dp_k}{dR_q} = \frac{dp_k}{dR_j} \quad , \text{ and } \quad \frac{dR_y}{dR_q} = \frac{dR_y}{dR_j} \quad .$$

$$(k = 1, 2, \ldots, n) \; (q, j, y = 2, 3, \ldots, m)$$

The right-hand side terms of the set of differentiation are not expected to remain the same under a change in different prices. Therefore, such equalities do not hold usually, and the variation in any price or interest rate is different for each price with respect to the change of which it is taken. The general solution above does not seem to provide any further information; but properties of the effects of a price change may be obtained by limiting the field of analysis with the necessary number of assumptions.

As a starting point, two sets of assumptions may be introduced.

(a) It is assumed that in the summation over all traders, the individual budget and purchasing power effects cancel out, leaving the total substitution effects only. This might occur, as the sign of the first two effects is not necessarily the same for each trader. In the preceding chapter it has been shown that the rise (fall) of a present market interest rate affects the demands in two ways: through the fall (rise)

which it produces in the prices of old securities and of common stocks, and through the increase (decrease) which it produces in the marginal utility of new fixed-yield securities. The first effect has the features of a budget effect, the second one is essentially a substitution effect. The present assumption imposes the condition that the summation of the first effect over all the individuals is equal to zero. J is then restricted to being the determinant in the summations of the individuals substitution effects alone.

(b) Such assumptions do not seem yet to provide any definite conclusions about the price derivatives, unless a restriction is imposed upon the market disequilibria described by the terms on the right-hand side of equations (5). The limitation will be that a variation in the price p_r, or in the interest rate R_q, provokes disequilibrium in the market for the commodity r or for the security q respectively, while all the other prices and interest rates change so as to leave all the other markets in equilibrium. Taken together with the first set of assumptions, this restriction will be shown to satisfy the criteria for imperfect and for perfect stability of the market, which Hicks states for an economy with no securities.[6] The Hicksian criteria are not unequivocal

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[6] J. R. Hicks, Value and Capital, pp. 248, 315.

conditions for stability. On the basis of dynamic results, Samuelson suggests that these criteria are neither necessary nor sufficient for stability;[7] however, arbitrary as they may be, such criteria seem to be sensible and reliable in most cases.

Mathematically, the limiting conditions on market disequilibria stated above are defined as

$$(6)\qquad \frac{dX_r}{dp_r} \neq 0\,, \quad \frac{dX_j}{dp_r} = 0\,, \quad \frac{dS_j}{dp_r} = 0\,, \quad (j = 2, 3, \ldots, m;\ j \neq r)$$
$$\frac{dS_q}{dR_q} \neq 0\,, \quad \frac{dX_h}{dR_q} = 0\,, \quad \frac{dS_y}{dR_q} = 0\,; \quad (h = 1, 2, \ldots, n;\ y \neq q)$$

The individual substitution terms are given by equations (15) and (24) of the third chapter. Forming a determinant in them, where they are in the same position in which the total substitution terms are in J, it is obtained

$$(7)\qquad \begin{vmatrix} \frac{\lambda U_{x_1x_1}}{U} & \cdots & \frac{\lambda U_{x_1x_n}}{U} & -u_{s_2}\alpha\frac{U_{x_1s_2}}{U} & \cdots & -u_{s_m}\alpha\frac{U_{x_1s_m}}{U} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\lambda U_{x_nx_1}}{U} & \cdots & \frac{\lambda U_{x_nx_n}}{U} & -u_{s_2}\alpha\frac{U_{x_ns_2}}{U} & \cdots & -u_{s_m}\alpha\frac{U_{x_ns_m}}{U} \\ \frac{\lambda U_{s_2x_1}}{U} & \cdots & \frac{\lambda U_{s_2x_n}}{U} & -u_{s_2}\alpha\frac{U_{s_2s_2}}{U} & \cdots & -u_{s_m}\alpha\frac{U_{s_2s_m}}{U} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{\lambda U_{s_mx_1}}{U} & \cdots & \frac{\lambda U_{s_mx_n}}{U} & -u_{s_2}\alpha\frac{U_{s_ms_2}}{U} & \cdots & -u_{s_m}\alpha\frac{U_{s_ms_m}}{U} \end{vmatrix}$$

Define the determinant $D = \left| \frac{U_{x_r s_j}}{U} \right|$. D and its principal minors are symmetrical and alternately opposite in sign as shown at page 73. Moreover, the second partial derivatives multiplying the terms in the determinant are all positive.

The solution for the market disequilibria, $\frac{dX_r}{dP_r}$ and $\frac{dS_j}{dR_j}$, is the summation over all the traders of the expressions

$$\frac{\lambda D}{D_{x_r x_r}}, \text{ and } -\frac{u_{s_j s_j} D}{D_{s_j s_j}} \qquad \begin{matrix} (r = 1, 2, \ldots, n) \\ (j = 2, 3, \ldots, m) \end{matrix}$$

which are respectively negative and positive. Consequently,

$$(8) \qquad \frac{dX_r}{dp_r} < 0, \text{ and } \frac{dS_j}{dR_j} > 0 \; . \qquad \begin{matrix} (r = 1, 2, \ldots, n) \\ (j = 2, 3, \ldots, m) \end{matrix}$$

The Hicksian criteria for imperfect stability read: let the price of the commodity r rise under the stated assumptions, then the market is imperfectly stable if the demand for the commodity r decreases, while all the other prices change so as to leave the other markets in equilibrium.[8] By analogy, the Hicksian assumptions can be extended to securities with this plausible restatement: the market is imperfectly stable, (1) when the price of the commodity r rises, (falls), if the

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[8] J. R. Hicks, Value and Capital, p. 248.

demand for the commodity r decreases (increases); (2) when the interest rate R_q rises (falls), if the demand for the new security q increases (decreases), while all other prices and interest rates change so as to leave the other markets in equilibrium. The first inequality establishes agreement between the hypotheses to which the model discussed here was subjected and the Hicksian conditions for imperfect stability of a market with commodities only. The second inequality satisfies the criteria for imperfect stability which were selected with regard to securities.

The Hicksian criteria for perfect stability in a market with no securities require that $\frac{dX_r}{dP_r}$ be negative when the prices of any number k of other commodities are adjusted so as to maintain equilibrium in the market for the same commodities, but all the remaining prices are unchanged.[9] In equations of the type of (5), limited to commodities and commodity prices, these criteria, rigorously interpreted, assume that (n - k - 2) terms on the left-hand side of each equation are equal to zero, and that k equations have a zero term on the right. Except for the one condition for perfect stability which is the same as the condition for imperfect stability (stating that all prices adjust themselves), the

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[9] J. R. Hicks, Value and Capital, pp. 248, 315.

Hicksian standards do not appear to be applicable, because, on the basis of the discussion of the general set (5), market disequilibria may assume any values. From Hicks' mathematical statement of the conditions, it appears that he implied that the markets of the commodities whose prices do not change are not taken into consideration at all. According to this interpretation, the criteria for perfect stability are seen to be the same as the criteria for imperfect stability in a market arbitrarily restricted to any number of commodities. They do not seem, therefore, to be very meaningful. Even such conditions can be extended to securities, by adding to criteria (1) and (2) the qualification that they should hold for arbitrary limitations of the market to any number of commodities and securities. Inspection of the determinant (7) reveals that the two sets of hypotheses, which have been set forth, satisfy also the Hicksian criteria for perfect stability of the market.

The price derivatives $\frac{dp_h}{dp_r}$ are equal to the product of $\frac{dX_r}{dp_r}$ and the summation over all the traders of the determinant ratio $\frac{D_{x_r x_h}}{\lambda D}$; the interest rates derivatives $\frac{dR_j}{dR_q}$ are equal to the product of $\frac{dS_q}{dR_q}$ and the summation over all the traders of $\frac{D_{s_j}}{-u_{s_j}\alpha D}$. From inequalities (8) and the symmetrical property of the determinant D recalled above, it is deduced that $\frac{dp_h}{dp_r}$ is of the same sign as $\frac{dp_r}{dp_h}$, $(h,r,=1,2,\ldots,n)$

and $\frac{dR_j}{dR_q}$ is of the same sign as $\frac{dR_q}{dR_j}$. $(j,q=2,3,\ldots,m)$ The following theorem has then been proved: under the two sets of restrictive assumptions envisaged above, the Hicksian criteria for perfect and imperfect stability, extended to securities in the way previously suggested, are met by the system. Under the same conditions, the effect of a rise (fall) in one price (rate of interest) on any other one is of the same sign as the effect of a rise (fall) in the latter price (rate of interest) on the former one. It is superfluous to point out that the present conclusions are a very particular case.[10]

2. The effects on the system of a change in the initial individual supplies, represented by differentiation of equations (3) and (4) will now be examined. From the assumption of perfect competition, it is known that a sufficiently large variation in the p's not to be safely neglected is only consequent to a cumulative change in a large number of individual supplies, which can be written $\frac{dp_j}{d\bar{X}_r}$. The differential $d\bar{X}_r$, however, is to be defined in terms of the

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[10] Hicks and Lange among others do not appear to recognize that, as the prices are the dependent variables, a change in one price, the explicit parameters being the same, should be explained with a variation in a new independent variable directly related to it as in the discussion of this section. J. R. Hicks, Value and Capital, p. 315. O. Lange, Price Flexibility and Employment, p. 92.

$d\bar{x}_r^{\gamma}$'s; and for every different combination of the $d\bar{x}_r^{\gamma}$'s composing $d\bar{X}_r$, there will be a correspondingly different dp_i, in general. These observations may be illustrated by carrying on the differentiation with respect to any combination whatsoever. Define, for example,

$$\beta = \frac{d\bar{X}_r}{\bar{X}_r} = \frac{d\bar{x}_r^{\gamma}}{\bar{x}_r^{\gamma}} = \frac{1}{\nu} \sum_{\gamma=1}^{\nu} \frac{d\bar{x}_r^{\gamma}}{\bar{x}_r^{\gamma}} \quad . \qquad (r = 1,2,\ldots,n)$$

β should be very small. Taking the differential of equations (3) and (4) with respect to the variation of $\bar{X}_r$, as defined by β, it is obtained

$$\bar{X}_r \left[\sum_{k=1}^{n} \frac{\partial X_i}{\partial p_k} \frac{dp_k}{d\bar{X}_r} + \sum_{q=2}^{m} \frac{\partial X_i}{\partial R_q} \frac{dR_q}{d\bar{X}_r} \right] = -\beta \bar{X}_r \epsilon_{x_r x_i}$$

$$\bar{X}_r \left[\sum_{k=1}^{n} \frac{\partial X_r}{\partial p_k} \frac{dp_k}{d\bar{X}_r} + \sum_{q=2}^{m} \frac{\partial X_r}{\partial R_q} \frac{dR_q}{d\bar{X}_r} \right] = -\beta \bar{X}_r \epsilon_{x_r x_r} + \beta \bar{X}_r$$

$$\bar{X}_r \left[\sum_{k=1}^{n} \frac{\partial S_j}{\partial p_k} \frac{dp_k}{d\bar{X}_r} + \sum_{q=2}^{m} \frac{\partial S_j}{\partial R_q} \frac{dR_q}{d\bar{X}_r} \right] = -\beta \bar{X}_r \epsilon_{x_r s_j} \quad ,$$

$$(i = 1,2,\ldots,r-1,r+1,\ldots,n) \quad (j = 2, 3, \ldots, m)$$

$\epsilon_{x_r x_i}$ representing the derivative $\frac{\partial X_i}{\partial \bar{X}_r}$, from the direct differentiation of equations (4). The amounts of the ϵ's

are the summations of the variations in the budget effects occasioned by the choice of different combinations. If the income effects could be neglected, then the ε's would be negligible and the determinant of the coefficients, J, would be sufficient to determine the variation in any price. In fact, solving under this assumption,

$$\frac{\partial p_i}{\partial \overline{X}_r} = \frac{J_{x_r x_i}}{J} \text{ , and } \frac{\partial R_j}{\partial \overline{X}_r} = \frac{J_{x_r s_j}}{J} \text{ . Yet, usually,}$$

$$\frac{\partial p_i}{\partial \overline{X}_r} = \frac{J_{x_r x_i}}{J} + \sum_{k=1}^{n} \varepsilon_{x_r x_k} \frac{J_{x_k x_i}}{J} + \sum_{q=2}^{m} \varepsilon_{x_r s_q} \frac{J_{s_q x_i}}{J} \text{ , and}$$

$$\frac{\partial R_j}{\partial \overline{X}_r} = \frac{J_{x_r s_j}}{J} + \sum_{k=1}^{n} \varepsilon_{x_r x_k} \frac{J_{x_k s_j}}{J} + \sum_{q=2}^{m} \varepsilon_{x_r s_q} \frac{J_{s_q s_j}}{J} \text{ , which}$$

are functions of the form $\frac{\partial p_i}{\partial \overline{X}_r} = h_i(\Delta_{\overline{X}_r}^{\gamma})$ and

$\frac{\partial R_j}{\partial \overline{X}_r} = h_j(\Delta_{\overline{X}_r}^{\gamma})$, where $\sum_{\gamma=1}^{\nu} \Delta_{\overline{X}_r}^{\gamma} = \Delta_{\overline{X}_r}$.

In the case in which the other total effects do not offset the total substitution effects, $\frac{J_{x_r x_r}}{J} < 0$, and the ε's, or the total budget effects caused by the choice of different combinations, would be so small as to make the terms

$$\sum_{k=1}^{n} \varepsilon_{x_r x_k} \frac{J_{x_k x_i}}{J} + \sum_{q=2}^{m} \varepsilon_{x_r s_q} \frac{J_{s_q x_i}}{J} \text{ ,}$$

etc., negligible. Then $\frac{dp_r}{d\overline{X}_r} < 0$, and $\frac{dp_i}{d\overline{X}_r} \simeq \frac{dp_r}{d\overline{X}_i}$, on the basis of the proof given at pages 126-127 under the same assumption. It can be concluded that, for a field of varia-

tion in the $\Delta^{\delta}_{x_r}$'s, i.e. for such a distribution among the consumers of the increase (decrease) of the supply of the good r, for which the total budget and purchasing power effects are relatively small, the price of that good will decrease (increase). Furthermore, for the same field of variations, the effect of an increase (decrease) in one commodity supply on the price of any other commodity is approximately equal to the effect of an increase (decrease) in the supply of the latter commodity on the price of the former one.

The same reasoning applies to the effects of a change in the total supply of any security. Consequently, the previous conclusions may be restated with regard to a change in the total supply of a security as follows

$$\frac{dR_j}{d\bar{B}_j} < 0 \quad \text{and} \quad \frac{dR_q}{d\bar{B}_j} \simeq \frac{dR_j}{d\bar{B}_q} \; . \qquad (j,q = 2, 3, \ldots, m)$$

In words, for a field of variation in the $\Delta^{\delta}_{s_j}$, for which the total budget and purchasing power effects are relatively small, the rate of interest will fall (rise) according to whether the total supply of the security increases (decreases). Similarly, within the same field of variations, the effect of an increase (decrease) in one security supply on the interest

rate of any other security is approximately equal to the effect of an increase (decrease) in the supply of the latter security on the interest rate of the former one. It is superfluous, perhaps, to stress the point that these results apply only to the very special case described.

The conclusions reached by the present discussion are seen to be more restrictive and less informative than the analogous ones found in the recent literature, which do not appear to be qualified too carefully for their limitation.

3. The aggregate effect of cumulative variations in a large number of utility functions remains to be analyzed. A shift in tastes, not accompanied by a change in the total supplies, must leave the total demands unchanged by virtue of the equilibrium conditions, and will only determine a tilting of the prices. Differentiating equations (3) and (4) with respect to any aggregate of functional variations defined as $\delta a = \sum_{\gamma=1}^{\nu} \delta u^{\gamma}$, the general result is

$$\sum_{h=1}^{n} \frac{\partial X_i}{\partial p_h} \frac{\delta p_h}{\delta a} + \sum_{q=2}^{m} \frac{\partial X_i}{\partial R_q} \frac{\delta R_q}{\delta a} = - \frac{\delta X_i}{\delta a}$$

$$\sum_{h=1}^{n} \frac{\partial S_j}{\partial p_h} \frac{\delta p_h}{\delta a} + \sum_{q=2}^{m} \frac{\partial S_j}{\partial R_q} \frac{\delta R_q}{\delta a} = - \frac{\delta S_j}{\delta a} \quad .$$

$$(i = 1, 2, \ldots, n) \; (j = 2, 3, \ldots, m)$$

The terms on the right-hand side of the equations are the

result of the direct differentiation of equations (4) in the u^{γ}'s. They have the same reason d'être which the u^{γ}'s have for being explicitly in the argument of equations (4). Equations (4) will in fact be consistent with a different set of p's and R's together with different utility functions. The solution for the unknowns, the price and interest rate changes, is given by

$$\frac{\delta p_k}{\delta a} = -\sum_{i=1}^{n} \frac{\delta X_i}{\delta a} \frac{J_{x_i x_k}}{J} - \sum_{j=2}^{m} \frac{\delta s_j}{\delta a} \frac{J_{s_j x_k}}{J} \qquad (k = 1, 2, \ldots, n)$$

which is of the form $\delta p_k = H(\delta^{\gamma}u)$, and similarly for which is of the form $\delta R_q = H_q(\delta^{\gamma}u)$.

It will aid clarity to investigate the result of the differentiation still further. Two effects of the cumulative variations in the utility functions can be traced: the direct effect on the demands for the commodities, the taste for which has shifted, that is always in the same direction as the shift; and the indirect effect on the demands for all commodities, provoked by the budget relations and by the equilibrium conditions. Where the preferences for various commodities shift, the two effects will be overlapping and they could be calculated on each commodity by considering the complex of reactions to each shift separately. The tilting in the prices necessary to bring the total demands to equality

with the total supplies again, is different according to which utility functions have varied and to the amount of the variation. A shift in the taste of traders who enjoy a large income will cause a larger variation in prices, ceteris paribus, than the same shift in the taste of low-income traders. Similarly, the shape of the shifting utility functions has an influence on the prices. These are the elements represented by the right-hand terms of the equations of differentiation.

The special case where the shift in the utility functions affects directly only one demand, say the demand for the h-th commodity, or the demand for the q-th security, provides the following interesting solution, for a field of variation of the u^{γ}'s, in which the budget and purchasing power effects do not offset the substitution effects,

$$\delta p_h = H_h(\delta^{\gamma} u) = - X_h \frac{J_{x_h x_h}}{J} \quad , \text{ or } \quad \frac{\delta p_h}{\delta x_h} = - \frac{J_{x_h x_h}}{J} > 0 \quad , \text{ and}$$

$$\frac{\delta R_q}{\delta s_q} = - \frac{J_{s_q s_q}}{J} < 0 \quad . \qquad \begin{matrix} (\gamma = 1, 2, \ldots, \nu) \\ (h = 1,2,\ldots,n) \\ (q = 2,3,\ldots,m) \end{matrix}$$

This solution, analogous to the one for the inverse effects indicated by equations (6), (7), and (8), is provided by the property established with regard to the determinant J under the same restriction. This result is the opposite of the one reached for the effect of an increase in a market

of interest on the total demand for the securities of the type governed by that rate. When the demand for one commodity (security) alone is directly increased (decreased) by a shift in tastes which provokes substitution effects stronger than the budget and purchasing power ones, the change in its price (interest rate) is positive (negative).[11]

Furthermore, $\frac{\delta p_i}{\delta X_\pi} = -\frac{J_{x_\pi x_i}}{J}$, and $\frac{\delta R_q}{\delta S_\gamma} = -\frac{J_{s_\gamma s_q}}{J}$. By the symmetrical property of J when the total budget and purchasing power effects are negligible, $\frac{\delta p_i}{\delta X_\pi} \simeq \frac{\delta p_\pi}{\delta X_i}$, $\frac{\delta R_q}{\delta S_\gamma} \simeq \frac{\delta R_\gamma}{\delta S_q}$ The effect of a rise (fall) in one commodity (security) total demand due to a shift in tastes on the price (interest rate) of another commodity (security) is approximately equal to the effect of a rise (fall) in the total demand for the latter commodity (security) on the price (interest rate) of the former one.[12]

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[11] Mosak concludes $\frac{\partial p_i}{\partial X_i} = \frac{J}{J_{ii}}$, but $\frac{\partial p_i}{\partial x_i} = \frac{J}{J_{ii}} < 0$ (The General-Equilibrium Theory of International Trade, fn. to p. 46). In his argument, an increase in the total demand for one commodity will lower its price, under the assumptions of fixed supplies and of predominant substitution effects. The present disagreement seems to arise from the fact that, while shifts in the demand curve are considered, Mosak examines movements along the demand curve itself.

[12] The equivalence maintained by J. R. Hicks (Value and Capital, p. 317) of the hypotheses of a change in one price with adjustments in all the others so as to leave their markets in equilibrium, and a shift in one demand, is shown to be erroneous.

C. Say's Law

In the study of the micro-model, it was proved that the definition of Say's law with respect to the individual violates the utility assumptions, and should, therefore, be rejected. The present section will be devoted to stating the relation between Say's law for the individual and the definition of a similar law for the market, and to analyzing the implications of the market law. Say's law for the market requires that the total demand for all commodities shall always be equal to their total supply, or that the total demand of purchasing power supplied to the commodity market shall always be equal to the amount demanded on that market. According to this assumption, no over-production is possible.

1. It is clear that in an economy with no securities at all the law for the individual is a sufficient and necessary condition for the market law, for the mathematical statement of the latter law

$$\sum_{i=1}^{n} p_i X_i = \sum_{i=1}^{n} p_i \bar{X}_i$$

is simply the summation of the budget restrictions over all individuals, and must hold in any case. Thus, in the absence of securities, both laws are true.

2. In an economy with securities, the two laws are not

equivalent.

a. Adding together the individual restrictions dictated by Say's law, there is obtained Say's law for the market. Consequently, the former law is a sufficient condition for the latter one.[13]

b. The necessity condition will be established only with regard to a special case: when all the preferences for the commodities should increase (decrease) together. The case of induced variations in the same direction in all commodity prices affords the same information under very restrictive assumptions.

(a) From property (4) at page 99, and the equilibrium equality between the value of the total commodity demands and the value of the total commodity supplies, it is easily seen that an increase (decrease) in the marginal utilities of all commodities for a large number of traders will cause an increase (decrease) in all the total commodity demands. In this case, Say's law for the market fails.

Say's law for the individual is expressed mathematically as a relation among the quantities demanded by the individual, added to the budget restriction, as indicated by equation (27) of the third chapter. Actually, this assumption represents

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[13] D. Patinkin, op. cit., p. 147.

a budget for the commodities, beside the usual budget. If the individual law should not contradict the utility assumptions, as it would occur if securities should not enter the utility function, property (3) on page 99 would apply, and from its summation over all the individuals it could be concluded that the value of all the total commodity demands would still be equal to the value of all the total supplies; and that Say's law for the individual would still hold. In the present case, Say's law for the individual is then seen to be a necessary condition for the market law. As the law for the individual is to be rejected, Say's law for the market is likewise rejected.

(b) The same conclusion can be reached for the case of an induced rise (fall) in all commodity prices, through changes in independent policies, etc., as explained elsewhere in this chapter. The problem, however, is to be qualified by three sets of assumptions: (1) the time-expectations of the parameters and of the variables must be equal to the present values; (2) the substitution effects must be predominant; (3) in correspondence to the induced variation in the price of each commodity, only the market for that commodity moves away from equilibrium. These strong limitations deprive the result of much significance. Thus, it does not appear to be convenient to stress the present case.

The proof of the solution stated, which is developed along the same lines as the preceding one, will not be given here.

(c) To the writer's knowledge, in any other case Say's law is not the statement of an existing condition, but it might be the recognition of an empirical repetition. It is clear that, while an exact occurrence of this assumption is very improbable,[14] its approximate realization might take place. By an identical reasoning as it was given for Say's law for the individual, it could be shown that the market law is equivalent to the condition of homogeneity of zero degree of the commodity demands in the prices. In the footnote to page 80, it was pointed out that such a mathematical condition reveals the presence of one more parameter than necessary. This parameter is the money price of the commodities, which can, therefore, assume any value as long as Say's law roughly holds. It is concluded that the market assumption designated as Say's law implies that the general level of prices is indeterminate, in the field where it holds as an empirical fact.[15]

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[14] D. Patinkin, op. cit., p. 148.

[15] O. Lange, Say's Law: a Restatement and Criticism, (in Studies in Mathematical Economics and Econometrics; Chicago: The University of Chicago Press, 1942), p. 64.

PART THREE

QUASI-DYNAMIC AND DYNAMIC MODELS

CHAPTER V

ECONOMIC PLANNING

A. Introduction

The models of economic relationships constructed in the previous chapters are the logical consequence of the qualification of the concept of utility such as to subject security holdings to the analysis of individual choice. This reformulation of the utility assumptions permits a more realistic approach than the one where utility does not extend to securities, and at the same time, by making possible the application of the same marginal method to security relationships as well as to commodity relationships, destroys the anomalous position which the classical monetary theory holds relative to the main body of the theory.

Although enjoying a higher degree of generality, the previous models were still presented in a static framework. The limitations of the static system become more apparent as more realistic hypotheses are set forth. In the present approach, the introduction of stocks of securities (and of stocks of commodities) diminishes greatly the usefulness of a static system. Indeed, the classical models, which did not contain securities and stocks of commodities carried over

from one period to the next, had no concern with future expectations.[1] But, as securities and stocks of goods constitute the link between the present and the future activity of the individual, a model including them implies the problem of future expectations. A system containing securities, which is of the static type and, hence, has no provisions for expressing the influence of future expectations explicitly, is affected by them through variations in the present utility of securities, i.e. through the medium of the structure of the utility functions. Thus, when the individual expectations are not represented directly in the system as independent variables, they affect the results by varying the utility function hand in hand with their shifts. The system consequently suffers strongly from frequent, uninformative variations in such parameters, unless expectations are restricted by a particular assumption, analogous to the one of stationary expectations which regulated the static system in this study, or to the one described in footnote 1 of this chapter.

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[1]This is not quite true. In the cases where the classical models introduce durable goods into the argument of the utility function, they are actually dependent on time-expectations. In such cases the elasticities of time expectations with respect to the present data are often implicitly assumed to be equal to unity, i.e. changes in present data are assumed to induce proportionate variations in the future expectations of the same data.

The individual holds money and other securities for three fundamental, distinct purposes: their services in facilitating transactions, in meeting contingencies, and in regulating the size of the future budgets, that is the service of speculation *in sensu lato*. A static analysis cannot distinguish among these services and isolate the speculation service as it would be desirable in order to develop the analysis of the impact of the interest rates on the system. Statics can only state that the services of securities all lumped together yield a definite satisfaction to the individual and that the relative maximization of utility determines the quantities of securities demanded, without any distinction among their services. The application of static analysis to stocks of money and interest-yielding securities is logically possible, as it has been shown, and offers positive results, but does not give all the significant information that is sought, in view of the fact that it does not put in relief those time elements, such as the speculative purpose and the far-reaching impact of future expectations on the system, which play a very important role in economic choice.

The shortcomings of static analysis may be easily overcome by adopting a concept of utility relationships which is at the same time more realistic and more fertile in the field of dynamic relationships. Utility is now made dependent not

only upon the present quantities demanded, as before, but also upon the quantities of commodities and securities expected to be demanded by the individual in the future. Where no securities are held by the trader and no amounts of commodities are carried over between two periods, the utility function, extended over expectations, does not give different results than the utility function studied in the third chapter. The superiority of the former function over the latter, in a system with securities, is set into light by intuitive and analytical considerations.

The system comprising time-expectations, which will now be examined, is instantaneous and refers to the present. As did the previous system, it is seen to cover an analytical period of time of arbitrary length, the commodities being purchased for consumption during it, and the rates of interest being computed on its length. At the beginning of each period, an instantaneous system is set up. Substantially, the new model is still static, because it serves to determine the actual, instantaneous demands like the preceding one. It contains, however, potentially dynamic elements. Although there is no reason why the expected demands for one point of time should be the same as the actual demands at the same point of time, once the initial stretch of time between the present and that point has elapsed, even if the expectations

of all the parameters should be realized, nevertheless, the expected future demands may be taken as a rough indication of future actual demands. Furthermore, the introduction of future expectations into the untility function gives a higher flexibility and sensitivity to the system. Savings, i.e. security holdings, are determined by the expectations of prices, interest rates, and income, together with the providence of the individual expressed by the utility to him of future consumption and future holdings of securities. The analysis of security speculation, the only possible kind in the absence of stocks of commodities which would permit commodity speculation, is greatly developed by the expectations framework.

It has already been pointed out that by this method a distinction is easily traced between the services of securities related to liquidity and those pertaining to speculation, the latter being based on their properties of store of purchasing power and of yield of interest. The connection between security holdings and time-expectations is made clear by the consideration of an extreme case in which satisfactions would be indifferent to the individual. It appears that in such a case he would hold a very limited amount of money, strictly necessary for the transactions service only, and no other securities. In fact, the sum of money held would be

subtracted from the present expenditure in order to swell the future budgets, the size of which would not concern the individual, if he should not be interested in future expenditures.

In conclusion, the particular form of its time structure adds definite dynamic elements to the system, even if it remains substantially instantaneous. In order to be distinguished from the usual static models and from the truly dynamic formulations, the present system is called a quasi-dynamic system.

The quasi-dynamic system of the type described above does not embrace expectations over all points of future time, but only at separate points. It is then defined as a discontinuous representation which has the following characteristics: time is only present implicitly in the form of a parameter; expectations are dated at the end of periods of equal length which are analytical and do not coincide necessarily with economic, actual periods of any kind. Expectations usually are uncertain; the expectation of any future parameter or dependent variable is not a unique value, but is represented by a probability function which may not be normal. The shape of the function will be influenced by the tendency of the individual to undertake risks. The prudent consumer will always foresee the worst possibilities. The

probability functions of his price-expectations for the goods he intends to buy will be negatively skewed. Conversely, a positive skewness will characterize a gambler. The probability function itself may be uncertain and its shape may depend on another probability function; probability would then be represented by a functional. For simplicity, the device of which use was already made in the static model[2] will be applied here by substituting a definite expectation to the probability function or functional, which will be that particular value to which the individual would react in the same way as he does to uncertain expectations. For the price expectation of the consumer such a value will be usually higher than the mean probability, because the difference should compensate the advantage of certainty. That value is called the "representative" expectation.

Similar to the budget in the purely static models, the quasi-dynamic budget is composed of holdings of money and other securities and of the income derived from the interest earned on them and from the sales of goods and services. A sum of money borrowed may be considered a negative security. For the sake of simplicity, income from the sales of goods and services is subtracted from utility considerations. This

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[2]See p. 55 in the third chapter.

device, together with the device of leaving aside stocks of commodities, has the advantage of limiting the field without imparing the validity of the argument with regard to monetary relationships which are the aim of this study.[3]

B. Economic Horizons

The ensemble of the expectations of the demands for commodities and securities is called the economic plan. From direct experience it is apprehended that future expectations bear satisfaction for the individual at the present, which usually diminishes the farther in time the expectations are taken.[4] The question now arises: for about how long does the

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[3]The case of commodity stocks will be briefly examined in footnote 9 to p.158.

[4]P. N. Rosenstein-Rodan explains the fact that the individual foresees less important wants only for a short future period with the factor of uncertainty which is large in the far future, not with the principle of underestimation of future wants. He reports to share this belief with Jevons, Sidgwick, Wieser, Pantaleoni, Wicksteed, Knight and H. Mayer. J. Bentham, Boehm-Bawerk, Marshall and Edgeworth subscribe to the theory of underestimation of future wants. J. Fisher's time preference notion agrees with the latter approach. The two points of view, however, appear to be consistent. Indeed, common observation suggests that, even if the future satisfaction from the consumption of a certain quantity of a good is prospected to be as intense as the present satisfaction derived from the present consumption of the same amount, the present satisfaction derived from the expectation of the future consumption of the same amount of the good might be much smaller depending on the degree of impatience of the individual. ("The Role of Time in Economic Theory," Economica, N. S., Vol. 1, No. 1, Feb. 1934, pp. 77-97)

consumer plan? He may be more farsighted with regard to certain goods and securities and less farsighted with regard to others. Consequently, the plan is not of the same length in all the expected demands. The end-point of the plan relative to any good or security will be designated as the economic horizon of that good or security for the consumer. From the above remarks it follows that, in general, every good and security may have a different economic horizon. A distinction can be made between one general horizon and particular horizons. The general horizon is defined as the point of time which includes the minimum interval of time comprising all the dated expectations of the individual which affect his present decisions. A particular horizon refers to the expected demand for one commodity or security which influences present decisions, and is defined as the limit of that minimum period of time within which all such expectations take place.

Mainly, the economic horizons are determined by subjective, non-economic elements which condition the economic factors, but fall outside the scope of the present analysis. Psychological, historical, and environmental elements in a broad sense have the greatest weight. The most striking feature of all expectations, economic and non-economic, is their uncertainty. The less certain expectations are, the

less is there any use to plan far ahead. It is well known that during time of war, of inflation, or of unsettled economic or political conditions, a hand-to-mouth policy is more widespread among the people than it is during more stable times. It has been noticed also that in an unstable economy the people become economically "neurotic" and are very sensitive to any new change or rumor. This may occur because their near expectations tend to oscillate widely at the slightest stimulus and the counteracting effect of far expectations does not operate. Uncertainty is, then, a valid element for the determination of the horizons. The amount of uncertainty is inversely related to the length of the planning period. A minimum of uncertainty probably encourages long-term planning and allows for expectations to affect present decisions, and vice versa.[5]

In the opinion of the writer, economic horizons have not been satisfactorily defined as yet. For statistical purposes, more or less empirical and arbitrary definitions are undoubtedly suitable, but they seem to be meaningless for a deductive analysis.[6] An attempt will now be made at defining the

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[5]Rosenstein-Rodan and Hicks consider uncertainty as the principal element restricting the horizon. J. R. Hicks, Value and Capital, pp. 139-228; P. N. Rosenstein-Rodan, op. cit.

[6]A statistical definition was first attempted by J. Tinbergen ("The Notions of Horizon and Expectancy in Dynamic Economics," Econometrica, Vol. 1, No. 3, July 1933, p. 247).

horizons with mathematical rigor, for a system with n commodities and no securities. The following results can be readily extended to money and other securities. There is a well-known theorem for the maximization of the utility function which states that, if in the proximity of the zero quantity of any good, its marginal utility divided by its price is smaller than the marginal utility of expenditure, such a good will not enter the conditions for the restricted maximum and no amount of it will be demanded. This marks a point of discontinuity in the marginal demand. It is on such a point that the determination of the horizons is founded. Let u_{it} be the marginal utility of the expected demand for the good i at time t, p_{it} be its representative expected price, and λ_t be the expected marginal utility of expenditure, defined by the Lagrange multiplier connecting the utility function with the constraining relation of the expected budget, for the maximization.

The horizon for the good r is defined at the point of time τ_{x_r} by the relationships

$$\lambda_{\tau_{x_r}} = \frac{u_{x_r,\tau_{x_r}}}{p_{r,\tau_{x_r}}} \text{ , and } \quad \lambda_t > \frac{u_{x_r t}}{p_{rt}} \quad . \qquad (t = \tau_{x_r}+1,\ \tau_{x_r}+2,\ \ldots)$$

If the price of the good r is expected to be high at a certain point of time, it might well happen that the expected demand

for the good r will be equal to zero at that time, but will exist at later points of time, because

$$\lambda_{\tau'_{x_r}} = \frac{u_{x_r,\tau'_{x_r}}}{p_{r,\tau'_{x_r}}} \quad , \quad \lambda_{\tau'_{x_r}+1} > \frac{u_{x_r,\tau'_{x_r}+1}}{p_{r,\tau'_{x_r}+1}} \quad , \text{ and } \quad \lambda_t = \frac{u_{x_r t}}{p_{rt}} \quad ,$$

$$(\tau'_{x_r}+2 \leq t \leq \tau)$$

τ'_{x_r} may be called a minor horizon. Clearly, the same commodity has only one horizon τ_{x_r}, but any number of minor horizons τ'_{x_r}. The larger are the periods during which the expectations of the demand for a commodity are absent within the horizons, the less will the expectations weigh in the consumer's choice. The general economic horizon is defined at time τ by the inequalities

$$\lambda_\tau = \frac{u_{x_r \tau}}{p_{r\tau}} \quad , \quad \lambda_\tau \geq \frac{u_{x_k}}{p_k} \quad , \quad \lambda_t > \frac{u_{x_i t}}{p_{it}} \quad . \qquad (k \neq r)$$

$$(i = 1, 2, \ldots, n) \; (t = \tau+1, \tau+2, \ldots)$$

The general horizon marks the end of the individual plan for all commodities.

An empirical principle might help determine the actual length of the economic plan. If the consumer is planning to hold any amounts of securities at a certain future point of

time, the expectations of the demands for one unit of money worth of each one of such securities should not possess marginal utilities smaller than the marginal utility of the expected expenditure at the same time. The economic horizon, then, does not embrace a period of time shorter than the one during which the consumer presently plans to hold securities. In most cases, thus, the economic plan would cover a long stretch of time usually not greater than one's expected lifetime; sometimes, as in the case of an estate left for definite educational or charitable purposes, etc., the economic plan extends beyond the life of the individual.

C. The Equilibrium Position

In some way, the new hypotheses about the utility concept justified by the above remarks should be tested for their ability to yield further theoretical information about economic relationships. The utility function comprising the expectations of future demands takes the form

$$(2) \qquad u = u\left(x_{it}, \frac{\pi_{jt}{}^{e}{}_{jt}}{p_{i,t+1}}\right) \qquad \begin{matrix}(i = 1, 2, \ldots, n)\\ (j = 1, 2, \ldots, m)\\ (t = 0, 1, \ldots, T)\end{matrix}$$

where the subscript t represents points of time at equal intervals from the present time, $t = 0$, to time T which is the general economic horizon. Clearly, the quantities at

time Q are actually demanded; the quantities at any other time are the demand-expectations.[7]

In the argument of the static utility function, securities appeared to be weighted by the price level of the trading moment at which they are demanded. Actually, our models indicate that securities are purchased at one moment in order to perform their services at the next one, and should, therefore, be weighted by the (expected) price level at this next moment. The assumption of equality of time-expectations with the explicit parameters of the static system, which was imposed on the static framework, makes the two price levels equal; and either date may be used indifferently for the

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[7]A similar scheme of economic planning was suggested by G. Tintner, "The Theoretical Derivation of Dynamic Demand Curves," (Econometrica, Vol. VI, No. 4, October, 1938, pp. 375-380). In the same article Tintner sketched the mathematical solution for a continuous consumption plan. (op. cit., p. 380). See also O. Lange, "Say's Law: a Restatement and Criticism," pp. 67-68; J. L. Mosak, The General-Equilibrium Theory of International Trade, pp. 116 ff.; H. Makower and J. Marschak, op. cit.; J. Marschak, op. cit. The influence of past experience on future expectations was investigated by C. F. Ross, Dynamic Economics (Bloomington: The Principia Press, 1934), pp. 14 ff.; J. M. Keynes, The General Theory of Employment, Interest and Money, pp. 46 ff., 147 ff.; J. R. Hicks, Value and Capital, pp. 125 ff.; G. C. Evans, Mathematical Introduction to Economics, (New York, 1930); R. H. Whitman, "The Problem of Statistical Demand Techniques for Producers' Goods: an Application to Steel," (Journal of Political Economy, Vol. 42, No. 5, Oct. 1934, pp. 577-594).

price level. The fact that securities at any trading moment are to be weighted by the price level at the next moment cannot, however, be overlooked in the quasi-dynamic utility function, without greatly affecting the correctness of the results. Consequently, present prices do not appear in the argument of the utility function: this will be shown to be an important qualification in comparative quasi-dynamics.

The budgetary restrictions are written

$$(3)\qquad \sum_{i=1}^{n} p_{1t}x_{1t} + \sum_{j=1}^{m} \pi_{jt}s_{jt} = \sum_{i=1}^{n} p_{1t}\bar{x}_{1t} + \sum_{j=1}^{m} s_{j,t-1}(\pi_{jt} + \pi_j r_{j,t-1}) \cdot$$

$$(t = 0, 1, \ldots, \tau)$$

In order to avoid confusion with the time subscript, the principal or par value of the security j is written π_j, etc. The parameters of the system are the p_{it}'s, the π_{jt}'s, the π_j's, the r_{jt}'s, the $\bar{x}_{it}$'s, and the $s_{j,-1}$'s.[8] The shorter the analytical periods of the plan, i.e. the more frequent the

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[8] For a complete description of the budget restrictions, $\pi_j r_{jt}$ should be taken to represent the dividend yield of common stocks, d_{kt}, for this type of securities. Furthermore, to any class j of fixed-yield securities belong securities issued at any preceding point of time ω, each with its own rate of interest $R_j\omega$. For simplicity, only one security of type j is shown in the equation with rate of interest r_{jt}. Actually, there may be present several securities of the same type but issued at different times, bearing each a different rate of interest.

transaction moments in the market, as described by the methodological framework, the more closely does the system approach reality. As long as transactions are separated by periods of time of any length, the analysis may be classified as discontinuous quasi-dynamics. By bringing the length of the period to the zero limit, a continuous analysis is obtained.

The equilibrium conditions are the first order conditions for the maximization of utility under the constraining relations of the present and expected budgets. The expression to be maximized is

$$F = u(x_{it}, \frac{\pi_{jt} s_{jt}}{p_{1,t+1}}) - \sum_{t=0}^{T} \lambda_t \left[\sum_{i=1}^{n} p_{it}(x_{it} - \bar{x}_{it}) + \sum_{j=1}^{m} \left(\pi_{jt} s_{jt} - s_{j,t-1}(\pi_{jt} + \pi_j r_{j,t-1}) \right) \right] = \text{Max.} \tag{4}$$

$$(i = 1, 2, \ldots, n) \quad (j = 1, 2, \ldots, m) \quad (t = 0, 1, \ldots, T)$$

The first order conditions for a maximum are readily derived as

$$\frac{\partial u}{\partial x_{it}} - \lambda_t p_{it} = 0 \qquad \begin{matrix}(i = 1, 2, \ldots, n)\\ (t = 0, 1, \ldots, T)\end{matrix} \tag{5,1}$$

$$\frac{\partial u}{\partial s_{jt}} - \lambda_t \pi_{jt} + \lambda_{t+1}(\pi_{j,t+1} + \pi_j r_{jt}) = 0$$

$$\frac{\partial u}{\partial s_{jT}} - \lambda_T \pi_{jT} = 0 \qquad \begin{matrix}(j = 1, 2, \ldots, m)\\ (t = 0, 1, \ldots, T-1)\end{matrix} \tag{5,2}$$

$$(5,3)\quad \sum_{i=1}^{n} p_{1t}(x_{it} - \bar{x}_{it}) + \sum_{j=1}^{m} \pi_{jt} s_{jt} - s_{j,t-1}(\pi_{jt} + \pi_j r_{j,t-1}) = 0 \,.$$

$$(t = 0, 1, \ldots, \tau)$$

There are $n(\tau + 1)$ equations (5, 1), $m(\tau + 1)$ equations (5, 2), $(\quad + 1)$ equations (5, 3); altogether $(n + m + 1)(\tau + 1)$ equations to determine n quantities demanded x_i for $(\tau + 1)$ trading moments, m quantities demanded s_j for $(\tau + 1)$ trading moments, and the quantity λ_t for $(\tau + 1)$ trading moments, which add up to $(n + m + 1)(\tau + 1)$ variables. The number of equations is equal to the number of variables: the system is therefore determined,[9] in general.

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[9]The more general approach with stocks of goods carried over may constitute a solid basis for a general theory of security and commodity speculation that would be well suited to supporting the further realistic assumption of uncertain prices. In this case, commodity assets, as well as securities, would perform speculation and contingency services. Holdings of commodity stocks for the purpose of facilitating exchanges appear to be rare in modern economies.

If commodity assets are introduced into the system, the equation of maximization reads

$$F' = u'(x_{1t}, y_{it}, \frac{\pi_{jt} s_{jt}}{p_{1,t+1}}) - \sum_{t=0}^{\tau} \lambda_t \left[\sum_{i=1}^{n} p_{1t}(x_{it} + y_{it} - \bar{x}_{it} - y_{i,t-1}) + \sum_{j=1}^{m} \left\{ \pi_{jt} s_{jt} - s_{j,t-1}(\pi_{jt} + \pi_j r_{j,t-1}) \right\} \right] = \text{Max.} \,.$$

$$(i = 1, 2, \ldots, n)\ (j = 1, 2, \ldots, m)\ (t = 0, 1, \ldots, \tau)$$

The interpretation of the group of first order conditions (5, 1) is the same as for the static analysis, except that the demands x_{it} for points of time different from zero are not actual quantities, but are expected quantities which might very well be different, when the point t of time becomes the present time, and the quantities x_{it} $(t = 1, 2, \ldots, T)$ are actually demanded.

The group of conditions (5, 2) offers a more informative

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y_{it} is the quantity of the commodity i demanded to be held as an asset. The necessary conditions for a maximum with respect to commodity assets are

$$\frac{\partial u}{\partial y_{it}} - \lambda_t p_{it} + \lambda_{t+1} p_{i,t+1} = 0$$

$$\frac{\partial u}{\partial y_i} - \lambda p = 0 \quad . \qquad \begin{matrix} (i = 1, 2, \ldots, n) \\ (t = 0, 1, \ldots, T-1) \end{matrix}$$

The partial derivatives in the above equations represent the marginal satisfaction derived from holding or expecting to hold any commodity stock for the services related to liquidity, of facilitating exchanges and providing against contingencies. The transaction service of a commodity stock seems to be negligible; the contingency service may be important. The marginal satisfaction derived from the speculation service is shown by the term $\lambda_{t+1} p_{i,t+1}$ for the demand y_{it}. The reader could very easily extend to commodity assets the observations which will be made about the first order conditions with respect to securities.

expression than in the static analysis, so as to make the present system a more satisfactory set-up for monetary analysis than the sheer static one. It permits a distinction between the marginal utilities of the services of securities related to liquidity, given by the derivatives $\frac{\partial u}{\partial S_{jt}}$, and the marginal utility of the service of speculation, based on their property of store of purchasing power and of yield of interest, which is furnished by the terms $(\pi_{jt}+\pi_j r_{jt})\lambda_{t+1}$ Rates of interest appear in all first order conditions derived with respect to interest-bearing securities, except for the conditions at the horizon.

These conditions admit of a literary description. The marginal utility of the actual or expected demand for money, for instance, at one point of time, is equal to the marginal utility of expenditure at the same point minus the marginal utility of expenditure at the next analytical point. The latter marginal utility is derived from the possession of money at the next period and is, thus, the marginal utility of speculation, for that money provides a larger amount of purchasing power to future budgets. A quantity of money is demanded such as to equate the total of the marginal utilities of speculation and of the services related to liquidity to the marginal utility of expenditure. Consequently, the marginal satisfaction derived from holding or expecting to

hold money at any time is the marginal satisfaction derived from liquidity. If the marginal utilities of expenditure at the two contiguous points of time were equal, the marginal utility of money would be null and so much money would be held for speculation purposes and for the services related to liquidity as to let the individual derive all possible utility from the latter services, i.e. as to make their marginal utility equal to zero. An analogous reasoning is applicable to other securities.

From repeated substitution in equations (5, 2), the following expression is derived for money

$$\lambda_t = \sum_{h=t}^{T} \frac{\partial u}{\partial s_{1h}} \qquad (t = 0, 1, \ldots, T)$$

and for any other security there is an analogous summation, complicated by the product of its yield and price at different points of time. Such summations mean that the marginal satisfaction derived from expenditure at any one point of time t, λ_t, is equal to the sum of the marginal satisfactions derived from expecting to hold money at every trading moment, from t to the end of the plan. Evidently, if the total marginal satisfaction derived from expecting to hold money at all moments after time t is larger than the marginal satisfaction from expecting to consume a dollar's worth of

commodities at the same time t, then the individual would expect to hold more money; and vice versa. As the other securities bear interest, this interest yield is compensated, in the equilibrium conditions, by the fact that the total marginal utility of expecting to hold any interest-bearing security from time t to the end of the plan is smaller than the marginal utility of expenditure at time t.

A very useful simplification would be attained if the budget restrictions could somehow be added together.[10] Unfortunately, this is not possible, for usually there are several kinds of interest-bearing securities, beside money, each with a different rate of interest going on the market. Moreover, even if there were only one kind of interest-bearing security beside money on the market, no way is found to establish a relation between the marginal utilities of expected expenditure, represented by the Lagrange multipliers, at any two different points of time. For particular purposes, therefore, a specific rate may be chosen as the general rate of discount, but no general theoretical rate is to be found by whose means it would be possible to establish an exact

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[10]This simplification was suggested by G. Tintner, "The Theoretical Derivation of Dynamic Demand Curves," p. 375; H. Makower and J. Marschak, op. cit.; J. L. Mosak, The General-Equilibrium Theory of International Trade, p. 117.

relation among the marginal utilities of expenditure at different points of time. The budget relations are to be kept separate and they appear in the maximization together with as many Lagrange multipliers as the number of expected trading moments.

The relationship between the λ_t's, i. e. the marginal utilities of (actual) expected expenditure at any two adjacent trading moments is made clear by the following considerations. Divide the equation containing the term $\frac{\partial u}{\partial s_{jt}}$, in the set (5, 2) by $\lambda_{t+1}\pi_{jt}$. The result is

$$(6) \qquad \frac{\lambda_t}{\lambda_{t+1}} = \frac{\pi_{j,t+1} + \pi_j r_{jt}}{\pi_{jt}} + \frac{1}{\lambda_{t+1}\pi_{jt}} \frac{\partial u}{\partial s_{jt}} .$$

$$(j = 1, 2, \ldots, m)$$
$$(t = 0, 1, \ldots, T-1)$$

In order to exclude the expectation of speculative gain or loss in the security purchase, which is a complication superfluous to the present reasoning, interest rate expectations may be assumed to be the same at all points of time, or $r_{j0} = r_{j1} = \ldots = r_{jT}$. Under this hypothesis, equation (6) can be rewritten as

$$(7) \qquad \frac{\lambda_t}{\lambda_{t+1}} = 1 + r_{jt} + L_{jt} \qquad \begin{matrix} (j = 1, 2, \ldots, m) \\ (t = 0, 1, \ldots, \tau-1) \end{matrix}$$

L_{jt} is a ratio between the present satisfaction derived from the expectation at time t of the marginal convenience, or liquidity, from holding security j, and the marginal utility of the expenditure expectation at time $t+1$. Inspection of equation (7) suggests that L_{jt} is of the same order of magnitude as r_{jt}. If the interest rate of the security j should be large, it is conceivable that L_{jt} could be negative. A negative L_{jt} would signify that the holding of this security involves a cost or sacrifice, for its safe-keeping, etc. It does not appear to be possible, however, to determine the size of L_{jt}. In the classical case where securities do not enter the utility function, no liquidity element enters the marginal analysis, and $L_{jt} = 0$. The value of the ratio of the marginal satisfactions derived from the expected expenditures at times t and $t+1$ becomes then equal to $(1 + r_{jt})$. r_{jt} is then found to be the general rate of discount. The $(\tau + 1)$ anticipated budgets can be consequently reduced to one by the discount rate.

I. Fisher, who describes the classical case with no

liquidity elements, calls r_{jt} the "rate of time preference."[11] In the general case, the rate of time preference is represented by $(r_{jt} + L_{jt})$, and no general rate of discount can be found.

D. Comparative Quasi-Dynamics

The sufficiency conditions for a maximum will now be examined. They will be utilized, similarly to those of static analysis, for the purpose of constructing a comparative quasi-dynamic theory, which is the study of the laws of change of the system. It is clear that the marginal utility of the expectation of the demand for any good at any trading moment is related to the demand expectations for all goods at every trading moment. The sufficiency conditions are stated by requiring the following symmetrical determinant and all its principal minors to be alternate in sign,[12]

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[11] I. Fisher, The Theory of Interest (New York: The Macmillan Co., 1930), pp. 62, 104, 291; in an uncertain world, p. 224. Fisher briefly treats the elements of liquidity or convenience of securities at pp. 215-216, but he does not take these elements into consideration when he sets the rate of time preference equal to the rate of interest. This equality is also stated in his previous work: The Rate of Interest (New York: The Macmillan Co., 1907), p. 118; qualified for risk, p. 218.

As Makower and Marschak, who formulated a general theory of assets, do not have assets in the utility function, their results are along the classical lines and are not as general and complete as the ones reached in this study. H. Makower and J. Marschak, "Assets, Prices and Monetary Theory," op. cit.; J. Marschak, "Money and the Theory of Assets," op. cit.

[12] It was already indicated in footnote 8 to page 53 that these sufficiency conditions are not at all necessary, but

$$H = \begin{vmatrix} u_{x_{1o}x_{1o}} & u_{x_{1o}s_{1o}} & u_{x_{1o}s_{m\tau}} & p_{1o} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{s_{1o}x_{1o}} & u_{s_{1o}s_{1o}} & u_{s_{1o}s_{m\tau}} & \pi_{1o} & -(\pi_{11}+\pi_1 r_{1o}) & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ u_{s_m x_{1o}} & u_{s_m s_{1o}} & u_{s_m s_{m\tau}} & 0 & 0 & \pi_{m\tau} \\ p_{1o} & \pi_{1o} & 0 & 0 & 0 & 0 \\ 0 & -(\pi_{11}+\pi_1 r_{1o}) & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \pi_{m\tau} & 0 & 0 & 0 \end{vmatrix}$$

A variation in any parameter at any moment of the plan will affect the present and expected demands for all commodities and securities in all periods, even the ones preceding the time at which the parameter is to be varied. It is interesting to note which effects are occasioned in the demands by a change in each parameter.

1. Effects of variations in the supplies. The simplest

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they are so probable that they are treated as such in the economic literature. The semi-definite case is of difficult occurrence and can be safely neglected.

effect is the budget effect, caused by a change in the parameters $\bar{x}_{rt}$'s and $s_{j,-1}$'s which compose the budget without affecting the system otherwise. The total budget effect is positive, because total demand is to increase with total supply, but unfortunately the sufficiency conditions cannot determine the sign of the budget effect on each demand separately. In order to acquaint the reader with this analysis, the expressions in the determinants for the effects of changes in the parameters above upon the actual and expected demands will be briefly listed.

Let $U_{t,x_{ih}}$ be the co-factor of the bordering row (column) t and of the column (row) of the second differentiation with respect to x_{ih}, and let $U_{x_{ih}x_{rt}}$ be the co-factor of the term $u_{x_{ih}x_{rt}}$, etc.

a. A variation in $\bar{x}_{rt}$ has the following effects on the actual and expected demands,

$$(9)\qquad \frac{\partial x_{ih}}{\partial \bar{x}_{rt}} = -\,p_{rt}\,\frac{H_{t,x_{ih}}}{H}\,, \qquad \frac{\partial s_{jh}}{\partial \bar{x}_{rt}} = -\,p_{rt}\,\frac{H_{t,s_{jh}}}{H}\,,$$

$$\frac{\partial \lambda_h}{\partial \bar{x}_{rt}} = p_{rt}\,\frac{H_{t,h}}{H}\,. \qquad (i,r = 1, 2, \ldots, n)\quad (t,h = 0, 1, \ldots, \tau)\quad (j = 1, 2, \ldots, m)$$

The results of a unit change in B_t, the expected budget at time t, are clearly the same as above, divided through by p_{rt}.

b. Let $s_{j,-1}$ vary; hence

$$(10)\qquad \frac{\partial x_{ih}}{\partial s_{j,-1}} = -(\pi_{jo}+\pi_j r_{j,-1}) \frac{H_{o,x_{ih}}}{H}, \quad \frac{\partial s_{qh}}{\partial s_{j,-1}} = -(\pi_{jo}+\pi_j r_{j,-1}) \frac{H_{o,s_{qh}}}{H},$$

$$\frac{\partial \lambda_h}{\partial s_{j,-1}} = (\pi_{jo}+\pi_j r_{j,-1}) \frac{H_{oh}}{H} \; . \qquad \begin{matrix}(i = 1, 2, \ldots, n)\\ (q, j = 1, 2, \ldots, m)\\ (h = 0, 1, \ldots, T)\end{matrix}$$

2. The demand effects of a variation in one present or expected commodity price are far more complicated. Also in quasi-dynamics, is applied the assumption, advanced in static analysis,[13] that the second cross partial derivatives of utility with respect to a security and a commodity are equal to zero.

a. The determinant solution for the variation in the expected demand for the commodity i at time h, corresponding to a change in the present price of the commodity r, is found to be

$$(11)\qquad \frac{\partial x_{ih}}{\partial p_{ro}} = (x_{ro} - \bar{x}_{ro}) \frac{H_{o,x_{ih}}}{H} + \lambda_o \frac{H_{x_{ro}x_{ih}}}{H} \; .$$

$$\begin{matrix}(i, r = 1, 2, \ldots, n)\\ (h = 0, 1, \ldots, T)\end{matrix}$$

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[13] See pp. 53-54.

It shows respectively a budget and a substitution effect but no purchasing power effect. In fact, a change in a present price does not affect the real purchasing power of the securities demanded, as long as the price-expectations do not vary, for such securities are destined to be exchanged in the future. In other words, present prices do not appear in the argument of the utility function for the calculation of the real purchasing power of the securities demanded, and consequently they have no purchasing power effect. The variation in the actual or expected demand for a security corresponding to a change in the present price of a commodity is given by an analogous expression.

b. The variation in any demand caused by a change in the expected future price of a commodity is composed by the two terms indicated by equation (11), plus a purchasing power term. The expressions for the variations in the security demands at any point of time and in the marginal utilities of the actual and expected expenditures with respect to a present price can be, therefore, derived directly from the more general expressions describing the effects of the change in any expected commodity price. The determinant solution for a variation in a commodity demand with respect to an expected commodity price is as follows

$$\frac{\partial x_{ih}}{\partial p_{rt}} = (x_{rt} - \bar{x}_{rt}) \frac{H_{t,x_{ih}}}{H} + \lambda_t \frac{H_{x_{rt}x_{ih}}}{H}$$

$$(12) \quad + \sum_{\omega=0}^{\tau} \sum_{q=1}^{m} \sum_{j=1}^{m} \sum_{l=1}^{m} \frac{\pi_{q\omega}\pi_{j,t-1}s_{j,t-1}}{p_{l,\omega+1}p_{rt}^2} \frac{\partial^2 u}{\partial \sigma_{jr,t-1} \partial \sigma_{ql\omega}} \frac{H_{s_{q\omega}x_{ih}}}{H}$$

$$+ \sum_{q=1}^{m} \frac{\pi_{q,t-1}}{p_{rt}^2} \frac{\partial u}{\partial \sigma_{qr,t-1}} \frac{H_{s_{q,t-1}x_{ih}}}{H} ,$$

$$(i,r = 1, 2, \ldots, n)$$

$$(h = 0,1, \ldots, \tau)$$

$$(t = 1, 2, \ldots, \tau)$$

where σ_{jit} is defined as being equal to $\frac{\pi_{jt}s_{jt}}{p_{i,t+1}}$, etc. A distinction is again to be made among three components in the total effect: in the succession in which they are formulated in equations (12), they are a budget, a substitution, and a purchasing power term. In order to simplify the notation, the second term on the right-hand side of equations (12), representing the substitution effect, is denoted by the symbol $(X)p_{rt}x_{ih}$, and the last two terms, representing the purchasing power effect, by $\sum_{\omega=0}^{\tau} \sum_{q=1}^{m} (S_{q\omega})p_{rt}x_{ih}$. Substituting equations (9) into equations (12), it is obtained

$$\frac{\partial x_{ih}}{\partial p_{rt}} = \frac{(\bar{x}_{rt} - x_{rt})}{p_{rt}} \frac{\partial x_{ih}}{\partial \bar{x}_{rt}} + (X) p_{rt} x_{ih}$$

(12,a)

$$+ \sum_{\omega=0}^{\tau} \sum_{q=1}^{m} (s_{q\omega}) p_{rt} x_{ih} \quad . \qquad \begin{matrix} (i,r = 1, 2, \ldots, n) \\ (h = 0, 1, \ldots, \tau) \\ (t = 1, 2, \ldots, \tau) \end{matrix}$$

Similarly, the variation in the actual or expected demand for any security with respect to a change in any expected commodity price is described by

$$\frac{\partial s_{jh}}{\partial p_{rt}} = \frac{(\bar{x}_{rt} - x_{rt})}{p_{rt}} \frac{\partial s_{jh}}{\partial \bar{x}_{rt}} + (X) p_{rt} x_{ih}$$

(13)

$$+ \sum_{\omega=0}^{\tau} \sum_{q=1}^{m} (s_{q\omega}) p_{rt} s_{jh} \quad . \qquad \begin{matrix} (r = 1, 2, \ldots, n) \\ (j = 1, 2, \ldots, m) \\ (h = 0, 1, \ldots, \tau) \\ (t = 1, 2, \ldots, \tau) \end{matrix}$$

The three components of the total effect have the same features for a security demand as for a commodity demand.

The marginal utility of the actual or expected expenditure is subject to the following variation, after a change in p_{rt},

$$\frac{\partial \lambda_h}{\partial p_{rt}} = \frac{(x_{rt} - \bar{x}_{rt})}{p_{rt}} \frac{\partial \lambda_h}{\partial \bar{x}_{rt}} + \frac{\lambda_t}{p_{rh}} \frac{\partial x_{rt}}{\partial \bar{x}_{rh}}$$

$$+ \sum_{\omega=0}^{\tau} \sum_{q=1}^{m} \sum_{j=1}^{m} \sum_{l=1}^{n} \frac{\pi_{q\omega} \pi_{j,t-1} s_{j,t-1}}{p_{l,\omega+1} p_{rt}^2 p_{rh}} \frac{\partial^2 u}{\partial \sigma_{jr,t-1} \partial \sigma_{ql\omega}} \frac{\partial s_q}{\partial \bar{x}_{rh}}$$

$$+ \sum_{q=1}^{m} \frac{\pi_{q,t-1}}{p_{rt}^2 p_{rh}} \frac{\partial u}{\partial \sigma_{qr,t-1}} \frac{\partial s_{q,t-1}}{\partial \bar{x}_{rh}} .$$

$$(r = 1, 2, \ldots, n)$$

$$(h = 0, 1, \ldots, \tau)$$

$$(t = 1, 2, \ldots, \tau)$$

The total effect on the marginal utility of expenditure at any time is composed by a sequence of budgetary terms. Properties of the substitution and purchasing power terms. The demand derivatives describe the shape of the demand functions in the proximity of the equilibrium point. The sufficiency conditions for the restricted maximum of the utility function impose certain properties on the budget, the substitution, and the purchasing power terms of the demand derivatives. They reveal, thus, laws of uniformity in the demand functions, which are the aim of the analysis of the individual behavior. Properties analogous to the ones indicated in the static model hold also in the economic plan. The first six properties refer to the substitution term. The third one also applies

to the budget and the purchasing power terms. The seventh rule is imposed on the three terms taken together.

(1) The first property is written

$$(X)p_{rt}x_{it} = (X)p_{it}x_{rt} \qquad \begin{matrix}(i,r = 1, 2, \ldots, n)\\(t = 0, 1, \ldots, T)\end{matrix}$$

from the property of symmetry of the determinant H, but, as in general $\lambda_t \neq \lambda_h$,

$$(X)p_{rt}x_{ih} \neq (X)p_{ih}x_{rt} \cdot \qquad \begin{matrix}(i,r = 1, 2, \ldots, n)\\(t,h = 0, 1, \ldots, T)\end{matrix}$$

Usually, however, if $t<h$, $\lambda_t > \lambda_h$, and

$$(X)p_{rt}x_{ih} > (X)p_{ih}x_{rt} \cdot \qquad \begin{matrix}(i,r = 1, 2, \ldots, n)\\(t,h = 0, 1, \ldots, T)\end{matrix}$$

The substitution effect of a change in an actual or expected commodity price upon the actual or expected demand for another commodity, at the same point of time, is equal to the substitution effect of a change in the price of the latter commodity upon the demand for the former commodity at the same time. Usually, a change in an actually or expected commodity price will have a larger substitution effect upon the expectation of any commodity demand, at a later time, than a change in the price of the latter commodity upon the demand for the former one.

(2) From the sufficiency conditions for the restricted maximum of the utility function, it is deduced, as in statics, that

$$(X)p_{rt}x_{rt} < 0 \quad \begin{matrix}(r = 1, 2, \ldots, n)\\ (t = 0, 1, \ldots, T)\end{matrix}$$

By the substitution effect, a rise (fall) in the actual or expected price of a commodity will decrease (increase) the actual or expected demand for that commodity, at the same time.

(3) The third property holds for the budget, the substitution, and the purchasing power effect. It is to be divided into four different formulations corresponding to four different cases. All such formulations are manifest to common sense.

(a) (i) The effect of a rise in any present commodity price on the total amount of purchasing power presently spent on all commodities and securities is equal to the difference between the present initial demand for the commodity whose price has changed, and the present supply of that commodity. The effect of a fall in the price is equal to the same difference taken with a negative sign. Such an effect is expressed by the equation

$$\sum_{i=1}^{n} p_{io} \frac{\partial x_{io}}{\partial p_{ro}} + \sum_{j=1}^{m} \pi_{jo} \frac{\partial s_{jo}}{\partial p_{ro}} = x_{ro} - \bar{x}_{ro} \quad .$$

$$(r = 1, 2, \ldots, n)$$

The proof of this theorem is identical to the one developed at page 71 for the analogous theorem in the static model. The proof is based on the observation that in the determinant solution for the equation above, the summations of the cofactors multiplied by the prices are the expansion of the determinant H for the budget term and the expansion of determinants with two equal rows for the other terms. The summations for the other terms are, therefore, equal to zero.

(ii) A change in the future expectation of any commodity price cannot affect the total amount of purchasing power presently spent on all commodities and securities. In mathematical terms

$$\sum_{i=1}^{n} p_{io} \frac{\partial x_{io}}{\partial p_{rt}} + \sum_{j=1}^{m} \pi_{jo} \frac{\partial s_{jo}}{\partial p_{rt}} = 0 \quad .$$

$$\begin{matrix} (r = 1, 2, \ldots, n) \\ (t = 1, 2, \ldots, T) \end{matrix}$$

The proof is along the same lines as the one for the previous theorem, with the difference that the summation of the cofactors multiplied by the prices are the expansion of a determinant with two equal rows for the budget term also.

(b) (i) A rise in a commodity price expected at any future time occasions a variation in the total amount of purchasing power expected to be spent at that time which is equal to the difference between the demand for and the supply of the commodity whose price has changed, expected at the same point of time, plus the variation in the amount of purchasing power of the securities supplied at that point of time. This effect is described by the equation

$$\sum_{i=1}^{n} p_{it} \frac{\partial x_{it}}{\partial p_{rt}} + \sum_{j=1}^{m} \pi_{jt} \frac{\partial s_{jt}}{\partial p_{rt}} = x_{rt} - \bar{x}_{rt} + \sum_{j=1}^{m} (\pi_{jt} + \pi_{j} r_{j,t-1}) \frac{\partial s_{jt}}{\partial p_{rt}} .$$

$$(r = 1, 2, \ldots, n)$$

$$(t = 1, 2, \ldots, T)$$

The present theorem is proved by noticing that the three summations in the equation, expressed in determinant form, are the expansion of H divided by itself for the budget term, and the expansions of determinants with two equal rows for the other terms.

(ii) A rise in an actual or expected commodity price provokes a variation in the total amount of purchasing power expected to be spent at any other time in the future, which is equal to the variation in the amount of purchasing power of the securities supplied at the latter point of time.

Mathematically,

$$\sum_{i=1}^{n} p_{ih} \frac{\partial x_{ih}}{\partial p_{rt}} + \sum_{j=1}^{m} \pi_{jh} \frac{\partial s_{jh}}{\partial p_{rt}} = \sum_{j=1}^{m} (\pi_{qh} + \pi_{q} r_{q,h-1}) \frac{\partial s_{j,h-1}}{\partial p_{rt}} \quad .$$

$$(r = 1, 2, \ldots, n)$$
$$(h = 1,2,\ldots,t-1,t+1,\ldots,T)$$
$$(t = 0, 1, \ldots, T)$$

For a proof, it could be easily shown that the cofactors in the solution for the present equation are the expansion of determinants with two equal rows, and are, therefore, equal to zero, for each component effect.

(4) Two theorems for the substitution term are analogous to the fourth rule of the static system.

(a) A proportionate variation in the prices of any group of commodities, actual or expected at one future time, will have a substitution effect in the opposite direction on the total amount of expenditure on that group of commodities at the same point of time. This is equivalent to writing

$$\beta \sum_{r=1}^{k} \sum_{i=1}^{k} p_{rt} p_{it}(X) p_{rt} x_{it} \text{ is opposite in sign to } \beta \; .$$

$$(k \leq n)$$

$$(t = 0, 1, \ldots, T)$$

β is defined as at page 71. The proof is similar to the one developed at pages 72-74, to which the reader is referred.

(b) A proportionate variation in all the expectations

of the price of one commodity from any point of time to another one in the plan will have a substitution effect in the opposite direction on the total amount of purchasing power expected to be spent on that commodity between the same two points of time. Therefore,

$$\beta \sum_{\omega=t}^{T} \sum_{h=t}^{T} p_{r\omega} p_{rh}(X) p_{r\omega} x_{rh} \text{ is opposite in sign to } \beta .$$

$$(t < T)$$

$$(t, T = 0, 1, \ldots, \tau)$$

By shifting columns and rows of the determinant H conveniently, it is obtained a form similar to the one of the preceding theorem, to which the same mathematical proof applies.

(5) The fifth property refers to the substitution effect, and is derived from the second and the third property. Similarly to the third property, it considers two formulations of the substitution term.

(a) The substitution effect of a change in the present price of a commodity upon the total amount of present expenditure on all other commodities and all securities is in the same direction as that change.

(b) The substitution effect of a rise (fall) in the price of a commodity expected at any future time, renders the variation in the total amount of purchasing power, expected to be spent on all other commodities and on all securities at

the same time, algebraically larger (smaller) than the variation in the amount of purchasing power of the expected supply of securities at that point of time.

(6) The substitution effect is subject to the sixth rule which is derived from the third and the fourth one. This rule is to be divided into three cases.

(a) (i) The substitution effect of a proportionate change in the present prices of any group of commodities upon the total amount of present expenditure on all other commodities and all securities is in the same direction as that change.

(ii) The substitution effect of a proportionate rise (fall) in the prices of any group of commodities, expected at any one future time, renders the variation in the total amount of purchasing power, expected to be spent on all other commodities and on all securities at the same time, algebraically larger (smaller) than the variation in the amount of purchasing power of the expected supply of securities at that point of time.

(b) In the substitution term, a proportionate rise (fall) in all the expectations of the price of one commodity between any two points of time in the plan will cause algebraically a larger (smaller) variation in the purchasing power expected to be spent on the other commodities, and on

the securities between the two points of time, than on the purchasing power of the expected supply of securities during the same period.

(7) The seventh property involves the budget, the substitution, and the purchasing power terms. It states a suggestive behavior of the demands, which appears to agree with common sense. A proportionate variation in all commodity prices at all points of the economic plan has approximately only a budgetary effect on an actual or expected commodity demand. It can be written in the form

$$\beta \sum_{t=0}^{\tau} \sum_{r=1}^{n} p_{rt} \frac{\partial x_{ih}}{\partial p_{rt}} = \beta \left[\sum_{t=0}^{\tau} \sum_{j=1}^{m} (\pi_{jt} s_{jt} - \pi_{j,t-1} s_{j,t-1}) \frac{1}{p_{rt}} \frac{\partial x_{ih}}{\partial \bar{x}_{rt}} - \sum_{h} \sum_{j=1}^{m} \pi_{jt}(X) p_{ih} s_{j\tau} \right] .$$

$$(i = 1, 2, \ldots, n) \quad (h = 0, 1, \ldots, \tau)$$

This expression is obtained by substituting in the determinant solution. The reader might very easily repeat the procedure for the result above, which will not be indicated here, for its length is not proportionate to the slight importance of the present topic. The effect on any security demand is only slightly different; the statement of its mathematical expression is superfluous. It will be sufficient to point out that the last term in the equation above representing the substitu-

tion effect is undoubtedly very small, and consequently the result is approximately a budget effect. The same observation holds also for any security demand. An analogous conclusion was reached for the static model. A proportionate rise (fall) in all prices causes a proportionate diminution (increase) in the purchasing power of the amounts supplied of securities. It is clear that there is to be a budget effect. The proportionate variation in the purchasing power of the securities leaves no room for a substitution effect. A substitution effect takes place only with respect to the amounts of securities which are expected to be demanded at the end of the economic plan.

3. The effects of variations in the rates of interest on the demands for commodities and securities will now be studied.

a. The reader is referred to pages 86-88 for the classifications of securities according to the type of structure presented by the functions of the security prices in the interest rates. In the quasi-dynamic framework, formulas (22) at page 88 read

$$\pi_{ht} = \left[\frac{r_h}{R_{ht}} + \left(1 - \frac{r_h}{R_{ht}}\right) \frac{1}{(1 + R_{ht})^{\tau_h - t}} \right] \pi_h \qquad (t = 0, 1, \ldots, T)$$

$$(h = 1, 2, \ldots, v)$$

(14)

$$\pi_{jt} = \frac{r_j}{R_{jt}} \pi_j \qquad (j = v+1, v+2, \ldots, w)$$

$$\pi_{kt} = \frac{d_{kt}}{R_{kt}} \qquad (k = w+1, w+2, \ldots, m)$$

Here, for each type of security h and j only one security is considered, issued at any time $\omega(\omega = t, t-1, \ldots \tau)$. r_h and r_j, then, stand respectively for $R_{h\omega}$ and $R_{j\omega}$. The system could be easily extended to more fixed-yield securities of the same type, issued at different dates, by means of an expansion in the notation. R_{ht} is the market rate of interest going at time t for bonds of type j which reach maturity at the same time as the bond whose price is considered. d_{kt} is the dividend yield of the corporate stock k at time t.

b. The differentiation of equations (14) with respect to an interest rate or to the dividend yield of a corporate stock yields results analogous to the ones stated at pages 89-90 for the differentiation of the static equations.

c. Finally, the variation is to be analyzed in the

actual or expected demand for a commodity x_{iq}, caused by a change in the actual or expected market rate of interest for consols of type j at time t ($t = 0, 1, \ldots, T$), R_{jt}. It has been already pointed out that, in order to simplify the analysis, only one security of type j, issued at time ω, appears in the budget equations. From equations (14) and the budget equations (3), it is manifest that a change in R_{jt} occasions a different variation in the security prices, in the budget, and consequently in the demands, according to whether $\omega < t$, or $\omega = t$. The two cases are better investigated separately.

(a) In the case of a change in the actual or expected rate of interest of consols of type j at a time t after the time ω of issue of the security of type j which is in the budget restrictions, the corresponding variation in the actual or expected demand for a commodity is given by

$$\frac{\partial x_{iq}}{\partial \pi_{jt}} \frac{\partial \pi_{jt}}{\partial R_{jt}} = \frac{1}{H} \left[(s_{jt} - s_{j,t-1}) H_{t,x_{iq}} - \lambda_t H_{s_{j,t-1} x_{iq}} \right.$$

$$- \lambda_{t+1} \frac{\pi_{j,t+1} + \pi_j r_{jt}}{\pi_{jt}} H_{s_{jt} x_{iq}}$$

$$\left. - \frac{s_{jt}}{\pi_{jt}} \sum_{\omega=0}^{T} \sum_{y=1}^{n} \frac{\partial^2 u}{\partial s_{jt} \partial s_{y\omega}} H_{s_{y\omega} x_{iq}} \right] \frac{\partial \pi_{jt}}{\partial R_{jt}} .$$

$$(i = 1, 2, \ldots, n)$$
$$(j = v+1, v+2, \ldots, w)$$
$$(q,t = 0, 1, \ldots, \tau)$$

By operating substitutions analogous to the ones indicated at page 92, and using equations (9) and (12,a), the reader may verify the following transformation

$$\frac{\partial x_{iq}}{\partial \pi_{jt}} \frac{\partial \pi_{jt}}{\partial R_{jt}} = \left[\frac{s_{j,t-1}}{p_{rt}} \frac{\partial x_{iq}}{\partial \bar{x}_{rt}} - \frac{s_{jt}(\pi_{j,t+1} + \pi_j R_{j\omega})}{\pi_{jt} p_{r,t+1}} \frac{\partial x_{iq}}{\partial x_{r,t+1}} \right.$$
$$\left. - \frac{\lambda_t}{\lambda_q} (X) p_{iq} s_{j,t-1} - \frac{\lambda_{t+1}(\pi_{j,t+1} + \pi_j R_{j\omega})}{\lambda_q \pi_{jt}} (X) p_{iq} s_{jt} \right] \frac{\partial \pi_{jt}}{\partial R_{jt}} .$$

$$(i = 1, 2, \ldots, n)$$
$$(j = v+1, v+2, \ldots, w)$$
$$(q,t = 0, 1, \ldots, \tau)$$

The effect of a change in a present or expected security price on the demand for a security is similar to the one on a commodity demand, except for the effect on the demand for the same security whose price varies. This last effect is described by the expression

$$\frac{\partial s_{jt}}{\partial \pi_{jt}} \frac{\partial \pi_{jt}}{\partial R_{jt}} = \frac{1}{H} \Bigg[- s_{j,t-1} H_{t,s_{jt}} - \lambda_t H_{s_{j,t-1} s_{jt}}$$

$$- \lambda_{t+1} \frac{\pi_{j,t+1} + \pi_j r_{jt}}{\pi_{jt}} H_{s_{jt} s_{jt}} - \frac{s_{jt}}{\pi_{jt}}$$

$$+ \frac{s_{jt}(\pi_{j,t+1} + \pi_j r_{jt})}{\pi_{jt}} H_{t+1,s_{jt}} \Bigg] \frac{\partial \pi_{jt}}{\partial R_{jt}} .$$

$$(j = v+1, v+2, \ldots, w)$$

$$(t = 0, 1, \ldots, \tau)$$

The total variation in a commodity demand or a security demand consequent to a change in an interest rate can be broken down into two income and two substitution components. The variation in the demand for a security of the type whose rate of interest changes reveals one more term, independent of the equilibrium position, of a negative sign.

(b) The going rate of interest for consols of type j at time ω, $R_{j\omega}$, is the rate of the consols of this type issued at the same time. As it was assumed that the consol of type j in the budget restrictions was issued at time ω, $R_{j\omega}$ is the rate of the consol, and its change will affect the equilibrium conditions directly and through the Π_{jq}'s ($q = \omega+1, \omega+2, \ldots, \tau$). Thus, there follows the result for

the induced variation in an actual or expected demand for a commodity subsequent to a change in $R_{j\omega}$,

$$\frac{\partial x_{iq}}{\partial R_{j\omega}} = \sum_{t=\omega+1}^{\tau} \frac{\partial x_{iq}}{\partial B_t} \frac{\partial B_t}{\partial R_{j\omega}} + \sum_{t=\omega+1}^{\tau} \frac{\partial x_{iq}}{\partial \pi_{jt}} \frac{\partial \pi_{jt}}{\partial R_{j\omega}} = \sum_{t=\omega+1}^{\tau} \frac{s_{j,t-1}\pi_j}{p_{rt}} \frac{\partial x_{ih}}{\partial \bar{x}_{rt}}$$

$$+ \sum_{t=\omega+1}^{\tau} \left[\frac{s_{j,t-1}}{p_{rt}} \frac{\partial x_{iq}}{\partial \bar{x}_{rt}} - \frac{s_{jt}(\pi_{j,t+1} + \pi_j R_{j\omega})}{\pi_{jt} p_{r,t+1}} \frac{\partial x_{iq}}{\partial \bar{x}_{rt}} \right.$$

$$\left. - \frac{\lambda_t}{\lambda_q} (X) p_{iq} s_{j,t-1} - \frac{\lambda_{t+1}(\pi_{j,t+1} + \pi_j R_{j\omega})}{\lambda_q \pi_{jt}} (X) p_{iq} s_{jt} \right] \frac{\partial \pi_{jt}}{\partial R_{j\omega}} .$$

$$(i = 1, 2, \ldots, n)$$

$$(j = v+1, v+2, \ldots, w)$$

$$(q, \omega = 0, 1, \ldots, \tau)$$

B_t indicates the budget at time t. The terms in brackets have been described above. The terms under the first summation are the usual budget terms which have been examined at page 167. Similar expressions represent the variations in the demands for securities.

d. The derivation of the demand effects following a change in the rates of interest of maturity bonds or in the dividend yields of common stocks, which are much of the same composition as the demand effects caused by a change in a consol rate, is left to the interested reader.

E. The Demand Functions

The quantities of commodities and securities demanded at present or expected to be demanded in the future, together with the marginal utilities of the present and expected expenditures, which are determined by the equilibrium conditions, are functions of all the given values or parameters of the system. The parameters are the same as described in the static functions, with the addition of their expectations at all future trading moments in the economic plan. Hence, they are the present and expected commodity supplies and commodity prices, the present and expected rates of interest and dividend yields of common stocks, the present supplies of the securities and the securities' principals or par values.

The study of the effects of variations in these parameters is the study of the structure and of the properties of the functions of the commodity and of the security demands, and of the marginal utilities of expenditure. As the static functions, so the ones derived from the economic plan constitute the basis for the construction of the market relationships. The quasi-dynamic demand functions, however, differ from the static ones in the fact that, while every static parameter either is common to all individual static functions (e.g. a price), or is summed up over all the individuals to

provide a market value (e.g. the individual supply of a commodity), the majority of the quasi-dynamic parameters (the time expectations) are neither common to various individual functions, nor their summations have a direct bearing on market relations.

If, at any time, a sufficiently large number of traders experience a shift in the same direction in their future expectations of prices or interest rates, a shift in the present market demands may be consequent, which would induce variations in the present prices and interest rates. Thus, expectations of future prices and interest rates affect present prices and interest rates; and it can be intuitively inferred that the relationship has a causality nexus also in the opposite direction.[14] Nevertheless, mass variations in the expectations should be suggested by further statistical hypotheses, or by observation aided by the economist's intuition.

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[14] A study of the interrelations between fluctuations of present prices and of price expectations was made by O. Lange, Price Flexibility and Employment, Ch. 5.

CHAPTER VI

A STATIC AND A DYNAMIC AGGREGATE SYSTEM

A. A Static Theory of Market Relations with Non-Stationary Time-Expectations

As well as the purely static micro-model, also the economic plan can be utilized for the construction of a static system of market aggregates, of the type presented in the fourth chapter. However, the assumption of stationary time-expectations, which was at the basis of the static analysis, was discarded in the economic plan. Consequently, the macro-system based on the economic plan will be subject to one more set of parameters: the individual time-expectations. With the introduction of these new parameters, the aggregate static system, developed from the static individual relations, is easily related to the individual economic plan.

A shift in any future expectation for a sufficiently large number of traders brings about variations in the present demands. Market variations, which are the sums of the individual variations, share the same properties with the individual variations. Such properties were studied in the preceding chapter. Furthermore, variations in the present market demands occasion identical effects on present prices and interest rates, if they are engendered by shifts

in future expectations, or, for instance, by shifts in the utility functions, of the kind that was studied in the third chapter. It is, thus, shown that movements in the prices and interest rates, produced by fluctuations in a group of future expectations, are easily traceable in the working of the previous models. Adequate hypotheses concerning the behavior of time-expectations of all the traders in the market cannot be supplied by the present theory of economic choice, and are to be suggested by the observations of economic facts.

B. A Dynamic Model of Market Relations

The conditions for dynamic stability of a market comprising securities beside commodities can very well be investigated by means of a dynamic model of the type studied by Samuelson and Lange,[1] At equilibrium, the static and the dynamic models are equivalent. Statics, however, differs from dynamics because it can study only certain kinds of variations, of a differential order. Its limitation is manifest, for instance, in the unsatisfactory way in which it treats a change in one price, a dependent variable, occasioned by an external shock. A dynamic model, on the other hand, describes the path in time of the adjustments to

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[1] P. A. Samuelson, op. cit., pp. 269-276; O. Lange, Price Flexibility and Employment, pp. 94-99.

the equilibrium position after a change, even large, in a parameter. In this sense, a dynamic model is more general and more informative than the static one.

The dynamic system which will be presented here is based on the hypotheses that the movements of prices and interest rates in time are functions of the excess demands. The excess demand for any good is defined as the difference between the demand for and the supply of the same good. From the static equations (1) of the fourth chapter,

$$(1) \qquad \begin{aligned} X_i &= X_i(p_r, R_q) \quad , \text{ and} \\ & \qquad\qquad (i,r = 1, 2, \ldots, n) \\ S_j &= S_j(p_r, R_q) \quad , \qquad (j,q = 2, 3, \ldots, m) \end{aligned}$$

under the condition that all the other parameters listed at page 116 and all time-expectations are constant. In the fourth chapter, it was made clear that the set of equilibrium prices and interest rates, together with equilibrium total demands, depend not only on the total amounts of commodities and securities supplied to the market, but also on the way in which such amounts are distributed among the traders, i.e. on the individual supplies. In this study, the quantities supplied were assumed to be constant. Such an assumption may still prove to be useful here, for it attributes a

"neutral" role to the supplies. Thereby, the conditions for market stability around the equilibrium position, which will be indicated in the present section, must be totally fulfilled by the demand derivatives. It might be objected that the assumption of constancy of the supplies imposes too heavy restrictions on the demand derivatives. As the theory of production is beyond the limits of this work, no attempt can be made at deriving from general principles rigorous laws of uniformity in the behavior of the quantities supplied relative to prices and interest rates, but conclusions about the supply behavior are to be accepted prima facie. The assumption of increasing costs and decreasing returns appears to be realistic and at the same time to furnish reasonable results in the general theory of production.[2] The opposite assumption does not seem to be justified by common observation and by the theory. It can be safely concluded, therefore, that, if the assumption of constant supplies is to be discarded, that would be in a direction which would ease the conditions for stability imposed on the demand behavior. The conditions which will result under such an assumption might be thought of as being more rigid than they actually would, where more realistic hypotheses on the supply behavior should be applied.

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[2]See footnote 1 on p. 111.

From equations (1) it appears that, in whatever way the time derivatives of prices and interest rates are related to the excess demands, they are ultimately functions of the prices and of the interest rates, or

$$(2)\qquad \begin{aligned} \frac{dp_i}{dt} &= G_i(p_r, R_q) \qquad (i,r = 1, 2, \ldots, n)\\ \frac{dR_j}{dt} &= F_j(p_r, R_q) \qquad (j,q = 2, 3, \ldots, m) \end{aligned}$$

Under the condition that the Jacobian of the set (1) is not equal to zero, equations (1) can be inverted and, by shifts in the origins of the demands, there is obtained

$$(3)\qquad \begin{aligned} p_i &= p_i(X_r - \overline{X}_r,\ s_q - \overline{s}_q) \qquad (i,r = 1, 2, \ldots, n)\\ R_j &= R_j(X_r - \overline{X}_r,\ s_q - \overline{s}_q) \qquad (j,q = 2, 3, \ldots, m) \end{aligned}$$

Consequently, in general from equations (2) and (3) it is deduced that each time derivative of a price or of an interest rate is a function of all the excess demands. Such functions are written as follows,

$$(4)\qquad \begin{aligned} \frac{dp_i}{dt} &= \overline{G}_i(X_r - \overline{X}_r,\ s_q - \overline{s}_q) \qquad (i,r = 1, 2, \ldots, n)\\ \frac{dR_j}{dt} &= \overline{F}_j(X_r - \overline{X}_r,\ s_q - \overline{s}_q) \qquad (j,q = 2, 3, \ldots, m) \end{aligned}$$

which, together with equations (1), determine the $\frac{dp_i}{dt}$'s and the $\frac{dR_j}{dt}$'s, the p_i's and the R_j's, in function of the excess demands. The excess demands are the explicit parameters and represent the amounts of disequilibrium of the system at any time t.

Samuelson advanced the hypothesis that the time variation of each price depends only upon the amount of disequilibrium in its market. This hypothesis can also be adopted for the security markets and expressed in the functional form

$$\frac{dp_i}{dt} = g_i(X_i - \bar{X}_i) \qquad (i = 1, 2, \ldots, n)$$

(5)

$$\frac{dR_j}{dt} = f_j(s_j - \bar{s}_j) \qquad (j = 2, 3, \ldots, m)$$

Samuelson imposes the following restrictions on the structure of the g_i functions, which will be extended to the f_j functions: $g_i(0) = 0$, $f_j(0) = 0$. It is known that, if the substitution effect is predominant, an increase in the demand for a commodity, occasioned by a positive shift in the preference for that good, brings about a rise in the price for that commodity.[3] For equilibrium, therefore, the following condition is added by Samuelson

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[3] See p. 136.

$$g_i'(0) > 0 \quad . \qquad (i = 1, 2, \ldots, n)$$

Similarly, the results attained on page 136 state that, for a predominant substitution effect, an increased total demand for a security, caused by a shift in the traders' preference, induces a fall in the present rate of interest going for that type of security. Consequently,

$$f_j'(0) < 0 \quad . \qquad (j = 2, 3, \ldots, m)$$

Economic reasoning would suggest that equations (4) are also more realistic, besides being more general, mathematically, than equations (5).[4] In fact, from functions (5) and the conditions on their structure, it is concluded that the price of a commodity is stable only when the market for that commodity is in equilibrium; and, vice versa, the market for

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[4] In Foundations of Economic Analysis, p. 274, Samuelson admits that a set of equations as given by (5) (with no securities and interest rates) is formally more general than a set as described by the first group of equations (5). In his previous article, however, he seemed to regard the particular case represented by the latter set to be founded on as general hypotheses as the static case for which Hicks stated his stability criteria. (P. A. Samuelson, "The Relation Between Hicksian Stability and True Dynamic Stability," Econometrica, XII, 1944).

a commodity is in equilibrium when the price of the commodity is stable. This does not appear to be true: the price of a commodity might vary so as to leave the market for that commodity in equilibrium, while other prices move so as to bring the other markets to equilibrium. Similar cases could be easily prospected. The reasoning above applies also to interest rates.

Yet, equations (4) would impose conditions of stability on the demands around the equilibrium position, which would not be very informative and would not admit a simple economic interpretation. Hence, here will be sought the stability conditions based on the assumption described by equations (5). The limitation of the assumption should not, however, be overlooked.

Expanding equations (5) at the equilibrium position, and retaining only the first power terms, it is obtained[5]

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[5] In an economy with no securities, the individual demands are homogeneous of zero degree in the commodity prices, and so are the market demands. Consequently, the first time derivatives of the prices are also homogeneous of zero degree in the prices. This means that, if all the prices varied in the same proportion, there is no tendency to further price changes, or that the price level is indeterminate. The requirement that the price of one commodity (the numeraire) be equal to one anchors the price level to a fixed value.

$$\frac{dp_i}{dt} = g_i' \sum_{r=1}^{n} \frac{\partial X_i}{\partial p_r} (p_r - p_r^o) + \sum_{q=1}^{m} \frac{\partial X_i}{\partial R_q} (R_q - R_q^o)$$

(6)

$$\frac{dR_j}{dt} = f_j' \sum_{r=1}^{n} \frac{\partial S_j}{\partial p_r} (p_r - p_r^o) + \sum_{q=1}^{m} \frac{\partial S_j}{\partial R_q} (R_q - R_q^o) \quad .$$

$$(i = 1, 2, \ldots, n)$$

$$(j = 2, 3, \ldots, m)$$

These expressions are an approximation to the values of the $\frac{dp_i}{dt}$'s and the $\frac{dR_j}{dt}$'s which is the closer, the smaller the value is. Noticing that the derivatives in the expansion, being taken at the points $X_i = \overline{X}_i$ and $S_j = \overline{S}_j$, are constants, the set (5) is seen to be a system of linear differential equations. The solution in the p_i's and the R_j's is

$$p_i(t) = \sum_{h=1}^{k} A_{ih}(t)\, e^{\mu_h t}$$

(7)

$$R_j(t) = \sum_{h=1}^{k} B_{jh}(t)\, e^{\mu_h t} \quad . \qquad \begin{matrix} (i = 1, 2, \ldots, n) \\ (j = 2, 3, \ldots, m) \\ (k \leq n+m-1) \end{matrix}$$

The A_{ih}'s and B_{jh}'s are polynomials in t of degree one less than the multiplicity of the root h. Calling $\frac{\partial X_i}{\partial p_r} = a_{ir}$, $\frac{\partial X_i}{\partial R_q} = b_{iq}$, $\frac{\partial S_j}{\partial p_r} = c_{jr}$, and $\frac{\partial S_j}{\partial R_q} = d_{jq}$, it is readily shown that μ_h $(h = 1, 2, \ldots, k;\ k \leq n+m-1)$ are the distinct roots

of the canonic equation

$$(8)\qquad F(\mu) = \begin{vmatrix} g'^{a}_{1\,11}-\mu & \cdots & g'^{a}_{1\,1n} & g'^{b}_{1\,11} & \cdots & g'^{b}_{1\,1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ g'^{a}_{n\,n1} & \cdots & g'^{a}_{n\,nn}-\mu & g'^{b}_{n\,n1} & \cdots & g'^{b}_{n\,nm} \\ f'^{c}_{1\,11} & \cdots & f'^{c}_{1\,1n} & f'^{d}_{1\,11}-\mu & \cdots & f'^{d}_{1\,1m} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ f'^{c}_{m\,m1} & \cdots & f'^{c}_{m\,mn} & f'^{d}_{m\,m1} & \cdots & f'^{d}_{m\,mm}-\mu \end{vmatrix}$$

if the canonic equation is not symmetrical, as it is to be expected from the influence of the income and purchasing power effects, some of the roots μ_h are complex, i.e. composed by an imaginary and a real part. The imaginary part occasions an oscillatory movement, because the number e, raised to an imaginary number multiplied by t, is equal to a harmonic in t. The condition for all the prices and interest rates to tend toward their equilibrium values is that all the real parts of the μ_h's be negative. This happens when the principal minors of the matrix of $F(\mu)$ are alternate in sign.[6]

The fundamental requirement for coordination and

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[6]See also O. Lange, Price Flexibility and Employment, pp. 95-96.

centralization for a general and consistent deductive analysis imposes now the question: is there exact correspondence between the stability conditions for the static and for the dynamic system, and how are they related? Such questions will be commented upon by the following observations,

(1) The hypotheses binding the two systems are not the same. The dynamic model which was chosen for the stability conditions is a particular case of the general one, as it was indicated at page195. The mechanism of adjustment of prices and interest rates is different in the particular and in the general model.

(2) The dynamic stability conditions are not placed on the static matrix (in the a, b, c, d coefficients in equation (8) alone, but on the matrix of (8) in the products of the static derivatives and the adjustment speeds. The Hicksian criteria for perfect stability which are the same as the dynamic conditions, however, are imposed only on the static matrix.

(3) In the static matrix, lack of symmetry does not hinder stability in the Hicksian criteria, but, in the matrix of the canonic equation (8) for dynamic stability, it brings about an oscillatory solution for the equilibrium position.[7]

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[7]Samuelson devised small matrices where the classical condition that the principal minors of the matrix of (8) be

It is concluded that the correspondence between the two sets of conditions is the stricter, the lesser weight the three differences stated above bear on the final solution.

The search of other dynamic systems and the study of the relation between their stability conditions and the ones already known is not pertinent to this argument. In this work, it suffices to have indicated that monetary analysis can be molded into a formal dynamic system of a familiar kind. It is obvious, however, that in so doing the system was slightly transformed. The time variations of the rates of interest are new variables introduced into the system; the property of homogeneity of zero degree of the rate of change in time of commodity prices in the commodity prices does not hold any longer; etc.

C. The Element of Uncertainty

Throughout this study, the element of uncertainty was not given direct attention. Its direct analysis was avoided

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alternate in sign is neither necessary nor sufficient for the real parts of the distinct roots of (8) to be negative. For a large number of variables, however, there appears to be a very large probability that the classical condition be both necessary and sufficient for the real parts of the distinct roots to be negative. Hence, for a large number of variables, the three points discussed above seem to exhaust the list of the hindrances to an exact correspondence between the dynamic and the static stability conditions. (P. A. Samuelson, Foundations of Economic Analysis, pp. 273-274)

by means of the devices of assuming stationary time-expectations and of considering "representative" expectations, i.e. those certain expectations to which the individual would react in the same way as to the actual uncertain ones.[8]

One significant conclusion can be drawn from this approach to uncertainty. The systems of monetary analysis presented here are consistent with the interpretation of representative expectations as being actual expectations in a world of certainty: for how important it might be, uncertainty of the future is not an essential constituent of monetary relationships.[9]

The mathematical study of uncertainty and risk is based on probability functions, the parameters of which might also be subject to a probability distribution. The very parameters of the latter distribution might have a probability distribution, and so on, ad libitum.[10] The insertion of such proba-

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[8] A. G. Hart criticizes the device of representative expected prices on the ground of its impracticality. (Anticipations, Uncertainty and Dynamic Planning, Studies in Business Administration of the University of Chicago, Vo. II, No. 1, 1941, p. 55). Representative expected parameters, however, do not seem to meet any objections as a theoretical device.

[9] This thesis is also implicitly supported by O. Lange, Price Flexibility and Employment, Chaps. 4-6.

[10] G. Tintner, "A Contribution to the Nonstatic Theory of Production," in Studies in Mathematical Economics and Econometrics, in Memory of Henry Schultz, pp. 98 ff.

bility functions into general-equilibrium analysis would definitely complicate it and would rather obscure the basic issues instead of yielding new information. It has been pointed out that an increase in uncertainty corresponds to a rise in the representative expected prices of the group of commodities with respect to which the trader is eminently a buyer, and to a fall in the representative expected prices of the group of commodities which the trader plans to sell.[11] An analogous statement can be made relative to the expectations of any other parameter. This important relationship is sufficient to indicate the reactions of the system constructed on representative expectations to changes in uncertainty. Such changes are represented by a shift in a group of representative expectations, which is to be defined by a different set of assumptions in general, for each particular situation. For instance, uncertainty usually increases the farther in the future the parameters are considered. According to the relationship outlined above, the representative expectations of the parameters become such as to discourage the demand expectations, the farther in the future they are. Uncertainty is consequently a fundamental cause for the limitation of future planning and for the determination of the economic horizon.

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[11] See p. 148.

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This dissertation was prepared under the direction of the Chairman of the Candidate's Supervisory Committee and has been approved by all members of the Committee. It was submitted to the Graduate Council and was approved as partial fulfilment of the requirements for the degree of Doctor of Philosophy.

July 23, 1949

J. M. Simpson
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